DOCUMENT RESUME

ED 222 341 SE 039 388

TITLE	School Mathematics Study Group, Unit Number Two. Chapter 3 - Informal Algorithms and Flow Charts. Chapter 4 - Applications and Mathematics Models.
INSTITUTION	Stanford Univ., Calif. School Mathematics Study Group.
SPONS AGENCY	National Science Foundation, Washington, D.C.
PUB DATE	68
NOTE	81p.; Document contains some light and broken type. For related documents, see ED 173 092-097.
EDRS PRICE	MF01/PC04 Plus Postage.
DESCRIPTORS	*Algorithms; *Flow Charts; High Schools; *Mathematical Applications; Mathematics Curriculum; Mathematics Education; Mathematics Instruction; *Secondary School Mathematics; *Textbooks
IDENTIFIERS	*School Mathematics Study Group

ABSTRACT

This is the second unit of a 15-unit School Mathematics Study Group (SMSG) mathematics text for high school students. Topics presented in the first chapter (Informal Algorithms and Flow Charts) include: changing a flat tire; algorithms, flow charts, and computers; assignment and variables; input and output; using a variable as a counter; decisions and branching; and flow charting the division algorithm. Topics presented in the second chapter (Applications and Mathematical Models) include: situations leading to geometric models; packing marbles (mathematical modeling exercise); and illustrations of mathematical models. (JN)

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SCHOOL MATHEMATICS STUDY GROUP

UNIT NUMBER TWO

Chapter 3. Informal Algorithms and Flow Charts

Chapter 4. Applications and Mathematical Models

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TABLE OF CONTENTS

Chapter 3

INFORMAL ALGORITHMS AND FLOW CHARTS

3-1. Changing a Flat Tire

You have experienced many times the need to follow instructions in order to carry out some process successfully. For example, putting together a model airplane, following a recipe, playing a new game, complying with the rules of conduct set down by your parents, are all instances where it is necessary to follow instructions in order to carry out a process.

Definition. A list of instructions for carrying out some process step by step is called an algorithm.

Most processes can be represented as algorithms in many different ways. Here is one algorithm for changing a flat tire.

Algorithm for Changing a Flat Tire

1. Jack up the car.
2. Unscrew the lugs.
3. Remove the wheel.
4. Put on the spare.
5. Screw on the lugs.
6. Jack the car down.

You may feel that we have not put enough steps into our algorithm. We have not considered getting the equipment out of the trunk, placing the jack, removing the hub-caps, loosening the lugs before jacking up the car, etc. These are good objections. Still, our list is good enough for getting across the idea of an algorithm. When we get to the stage of writing algorithms for mathematical processes we will have to be much more precise.

A flow chart is a diagram for picturing an algorithm. We will give a flow chart for our flat tire algorithm and then explain it.

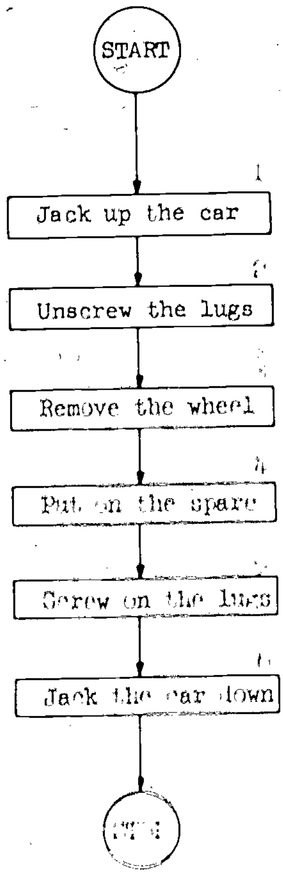


Figure 1. First flow chart for a flat tire.

In this flow chart, as in most, we see

and

.

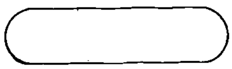
We also observe in our flow chart that each instruction is enclosed in a frame or box. A little later on we will see that the shape of the frame tells us what kind of instruction appears inside. Commands to take some action are written in rectangular frames.

In Figure 1 all instructions are of this form, so they all have rectangular frames.

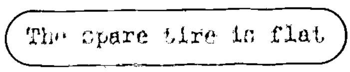
To carry out the process shown in a flow chart we go to START, follow the arrow to the first "box" and carry out the instruction given there, then follow the arrow to the next box, etc.

After drawing a flow chart we always look to see whether we can improve it. In the flat tire algorithm we forgot to check whether the spare was flat. Drivers seldom think at a service station to check the air in the spare tire, and sometimes it is flat when it is needed.

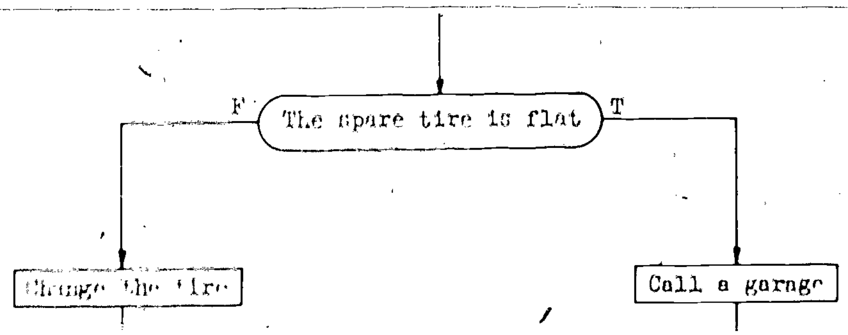
If the spare is flat then we certainly do not want to go to all the trouble of changing the tire. Instead we should call a garage. To make this decision we introduce a new kind of frame into our flow chart. The frame is oval in shape.



Inside this frame we find a statement on which we make a decision.



We have two exits from this box, one labeled T (true) and the other labeled F (false). After checking whether the statement is true or false, we leave the box at the corresponding exit and go on to the next box.



Such an oval box is called a decision box. When we put this flow chart fragment into our flat tire flow chart of Figure 1 we obtain the flow chart of Figure 2.

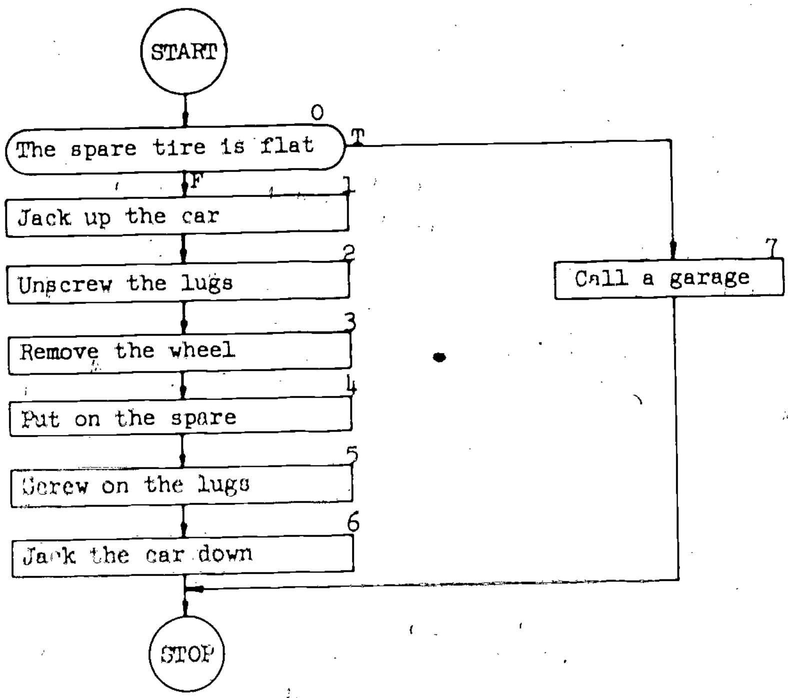
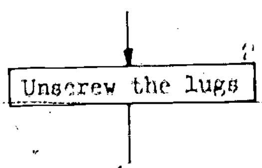
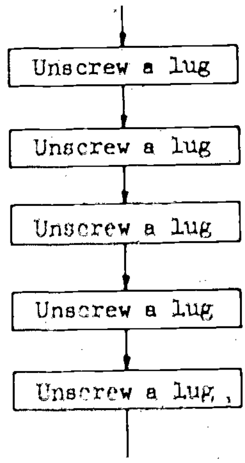


Figure 2. Second flow chart for changing a flat tire.

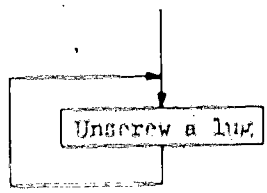
There is still one more improvement we would like to make on our flow chart. Let us look at box 2 in our flow chart.



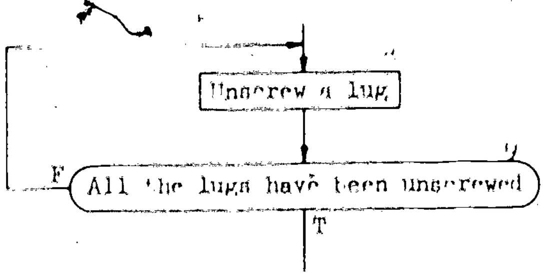
Actually this stands for a number of tasks, or rather the repeating of the same task. Since this automobile wheel has five lugs, one way of showing this is to have five frames.



This is correct but we can simplify this diagram by introducing a loop.



We see that when we leave this box we are sent right back to repeat the task. The trouble with this idea is that we have no way of getting out of the loop to the next task. We are caught in an endless loop. We can correct this situation by placing a decision box in our flow chart as shown in the figure below.



We get our final flow chart for changing a flat tire (Figure [illegible]) by replacing box [illegible] by boxes 8 and 9 and making a similar replacement for box [illegible].

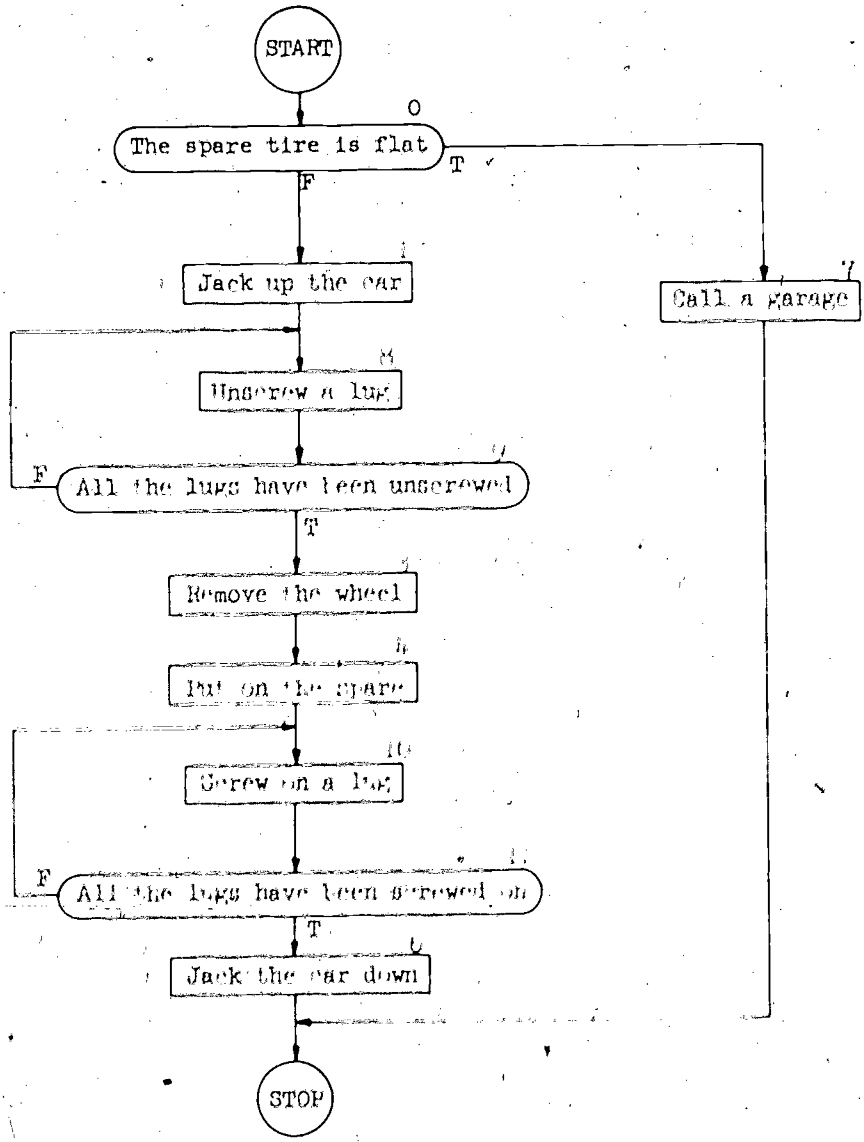

START
The spare tire is flat
T
F
Jack up the car
Call a garage
Unscrew a lug
F
All the lugs have been unscrewed
T
Remove the wheel
Put on the spare
Screw on a lug
F
All the lugs have been screwed on
T
Jack the car down
STOP

Figure 3

Exercises 3-1

(Class Discussion)

1. Mark's father's favorite breakfast consisted of orange juice, toast, milk, and a fried egg (sunnyside up) basted in butter. On Fathers' Day, Mark decided he would surprise his father by cooking breakfast for him but didn't know how to fry the egg. Mark's older sister (who was a computer programmer) constructed a flow chart showing Mark how to fry an egg the way their father liked it cooked.

   To start, Mark's sister listed the basic processes needed to cook an egg.

   1. Place frying pan on burner.
   2. Set heat under frying pan to medium-low.
   3. Put butter in pan.
   4. Break egg into frying pan.
   5. Baste egg with melted butter.
   6. Serve egg to Dad.

   (a) Construct the first flow chart for frying an egg. (See Figure 1, page .)

   (b) It is always possible that the family may be out of butter or eggs. Change your flow chart to account for this possibility by inserting a decision box in the correct place with the statement "We have both butter and eggs."

   (c) As there are two exits from a decision box, one labeled T (true) and one labeled F (false), the sister decided that if they were out of butter or eggs Mark's father would be served cold cereal. Change your flow chart to take account of this possibility.

   (d) Before breaking an egg into the frying pan the butter should be melted. Insert a decision box with the statement, "the butter has melted", into the flow chart.

   (e) If it is T (true) that the butter has melted, the next step would be to "break the egg into the frying pan". If the statement is F (false) then Mark must "wait $\frac{1}{2}$ minute". Using a loop and a rectangular box, change your flow chart to account for this possibility.

(f) To determine whether the egg is cooked or not, the white should not be transparent. If it is transparent, then more basting is needed. Change your flow chart by putting in a decision box, and loop, with the statement "the egg white is transparent".

With this last change, we get our final flow chart for frying an egg.

5-2. Algorithms, Flow Charts, and Computers

First you should know that our study of algorithms and flow charts is inspired by computers. Basically a computer does arithmetic: it can add, it can subtract, it can multiply, it can divide. The computer can perform these operations very rapidly. In fact, the computer can do millions of these arithmetic operations in a single second. By combining a vast number of simple arithmetical calculations a computer can solve in less than a minute a problem which might require weeks of hand computations. In an hour it can handle a problem needing years of hand computations.

The computer's great speed in performing an arithmetic operation would be of little value if after each of these calculations we had to stop and give the instruction as to what to do next. The time required to give the computer its millions of instructions would make it impossible for the computer to reduce computing time by much more than half. The secret is that all the instructions are put into the computer at the beginning of the problem so that the computer can get at these instructions using the same kind of speed which it uses on its arithmetic operations.

Still, if the computer is to perform millions of operations, will this require millions of instructions? Surely these would require an enormous amount of time to prepare. The answer is that as few as ten or twenty actual instructions may be used to tell the computer to make millions of calculations. The secret here lies in algorithms involving <u>repetition</u>. On a small scale we have met this idea in our flat tire algorithm in the preceding section when we introduced the loop in the flow chart. We passed through this loop several times before leaving it to perform another task. The instructions given a computer are in the form of algorithms, involving much looping, so that a small number of instructions can result in a large number of operations.

The first task in preparing a problem for a computer is to construct an algorithm for the problem. Usually it is not possible to see in advance just exactly what the computer will do at each step. We do know that if the instructions are followed the correct answer will be obtained. It may seem difficult to understand how we can give instructions for solving a problem but not know in advance what steps will be taken. The following exercises will illustrate this.

Exercises 3-2a

(Class Discussion)

1. Suppose a hiker is lost in the woods without a map or a compass. We will assume for simplicity that the man has found a river or stream. We want to construct an algorithm that will help the hiker find his way back to civilization so that he does not wander around aimlessly and never find a town. (We will assume that all rivers flow toward the sea.)

   (a) After the hiker has found a river what is the first command that you would give him?

   (b) If the hiker follows the river and comes to a town, then what command would you give him in the algorithm?

   (c) What would you instruct the hiker to do if the river flows into another river?

   (d) What would you instruct the hiker to do if the river flows into a lake?

   (e) What would you instruct the hiker to do if he comes to a river that flows out of the lake?

   (f) What would you instruct the hiker to do if the river flows into the sea?

   (g) Compare your answers with the following flow chart.

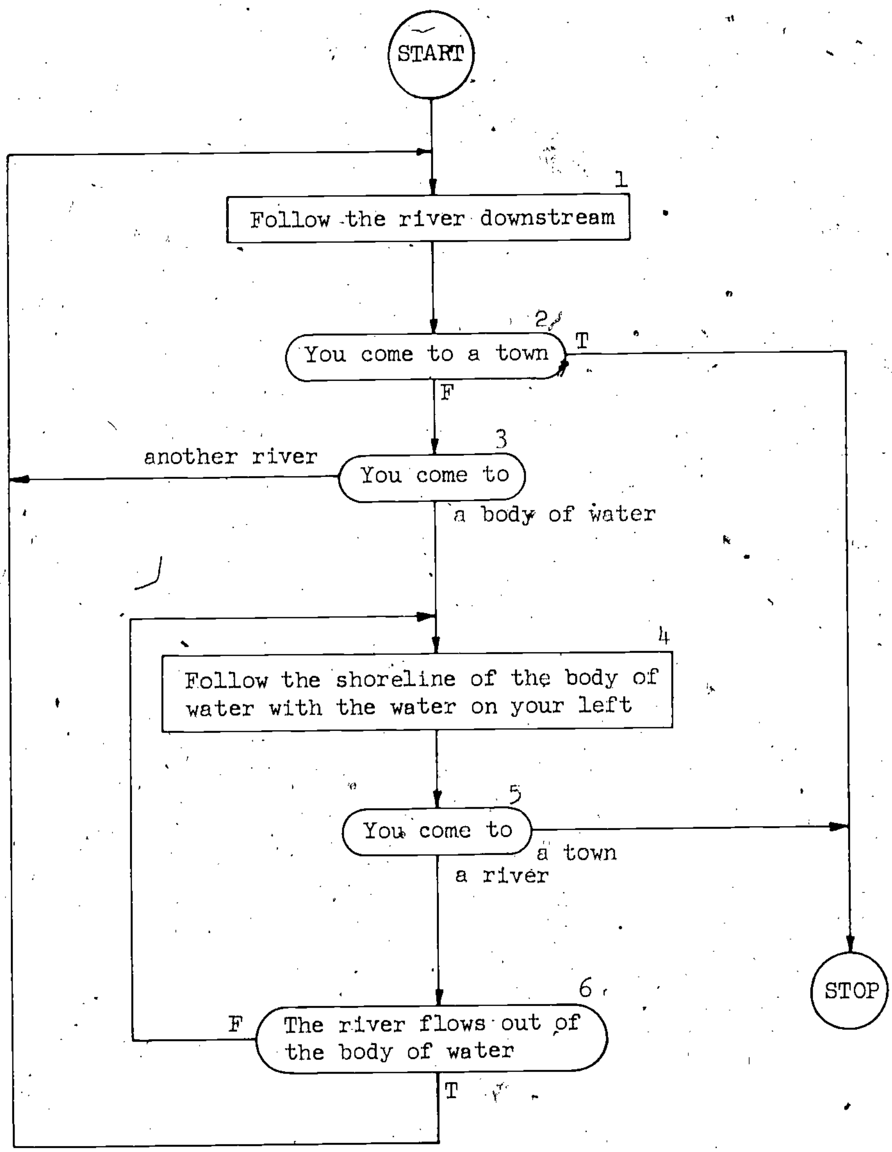


Figure 4. Lost in the woods.

As you can see, even though we have no idea what path the lost hiker will actually travel in following our flow chart, we believe that this flow chart will finally lead him back to civilization.

Of course, there are many different algorithms that could be constructed which would also lead the hiker back to civilization. Notice that in two of our decision boxes the exits are labeled with words rather than T or F. Also notice that the "body of water" referred to can be either the sea or a lake.

Exercises 3-2b

1. Two parties of hunters (A, B, C, D in one party, and X, Y in the other) on Gull Island became separated during the hurricane of 1964. Their positions after the storm are shown on the map in Figure 5. Each followed the flow chart of Figure 4 to find his way back to civilization.

   (a) For each hunter give the town he finally reached.

   (b) List the hunters in the order of the distance they traveled.

   (c) Which hunters from different parties reached the same town?

   (d) Which hunters from the same party arrived at towns that are the farthest apart?

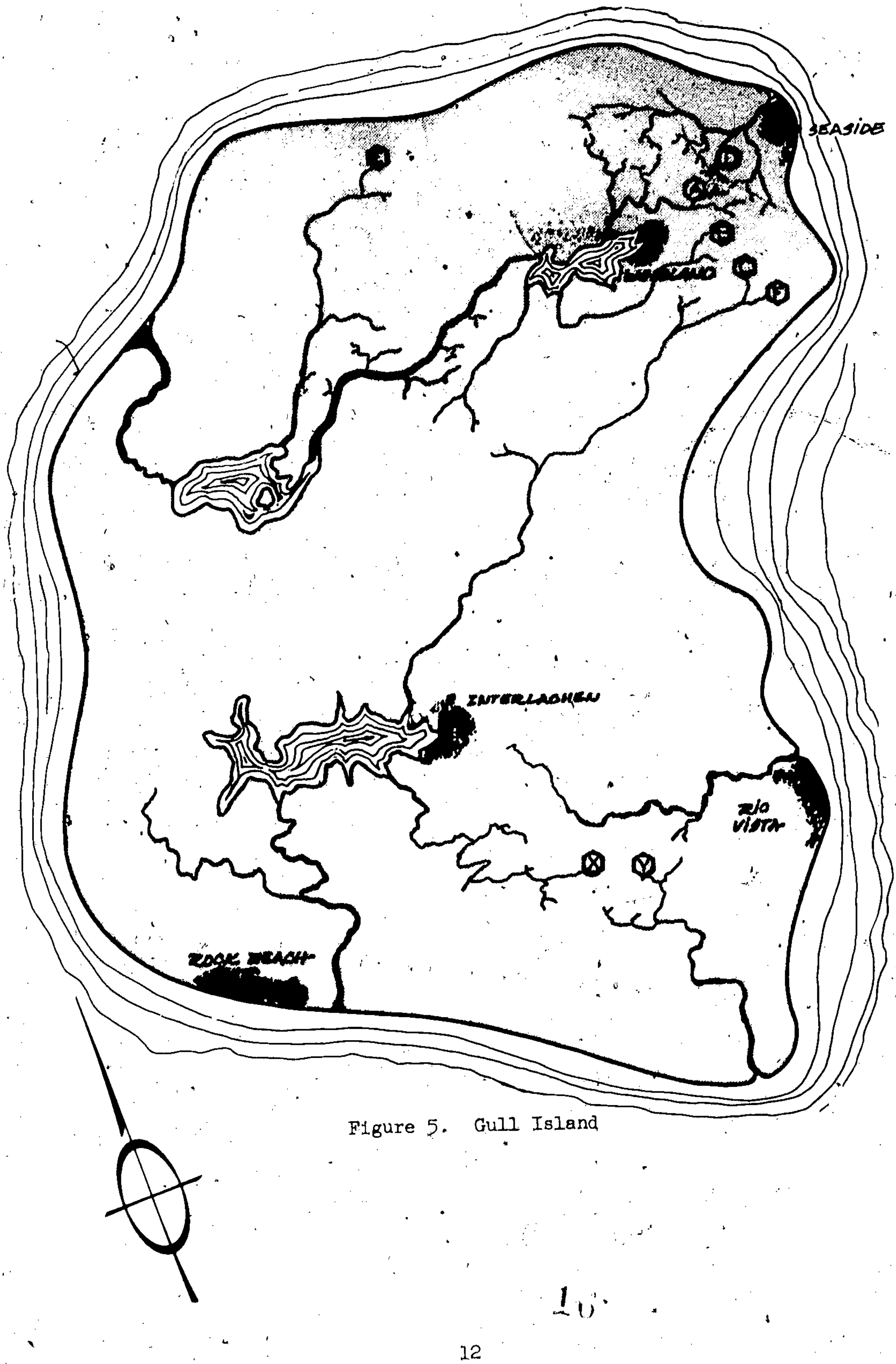


Figure 5. Gull Island

16

2. In 1768 Gull Island was a haven for pirates, and it is claimed by some that buried treasure still exists somewhere on the island. An old pirates' log book was found recently which indicated that the treasure was buried under an old oak tree at the headwaters of some stream on Gull Island. It is known that there are old oak trees at points A, B, C, D, E, F, X, and Y whose positions are shown on the map on the preceding page. The instructions given in the log book are shown in the following flow chart, Figure 6. Find the point where the treasure hunters should dig for the treasure.

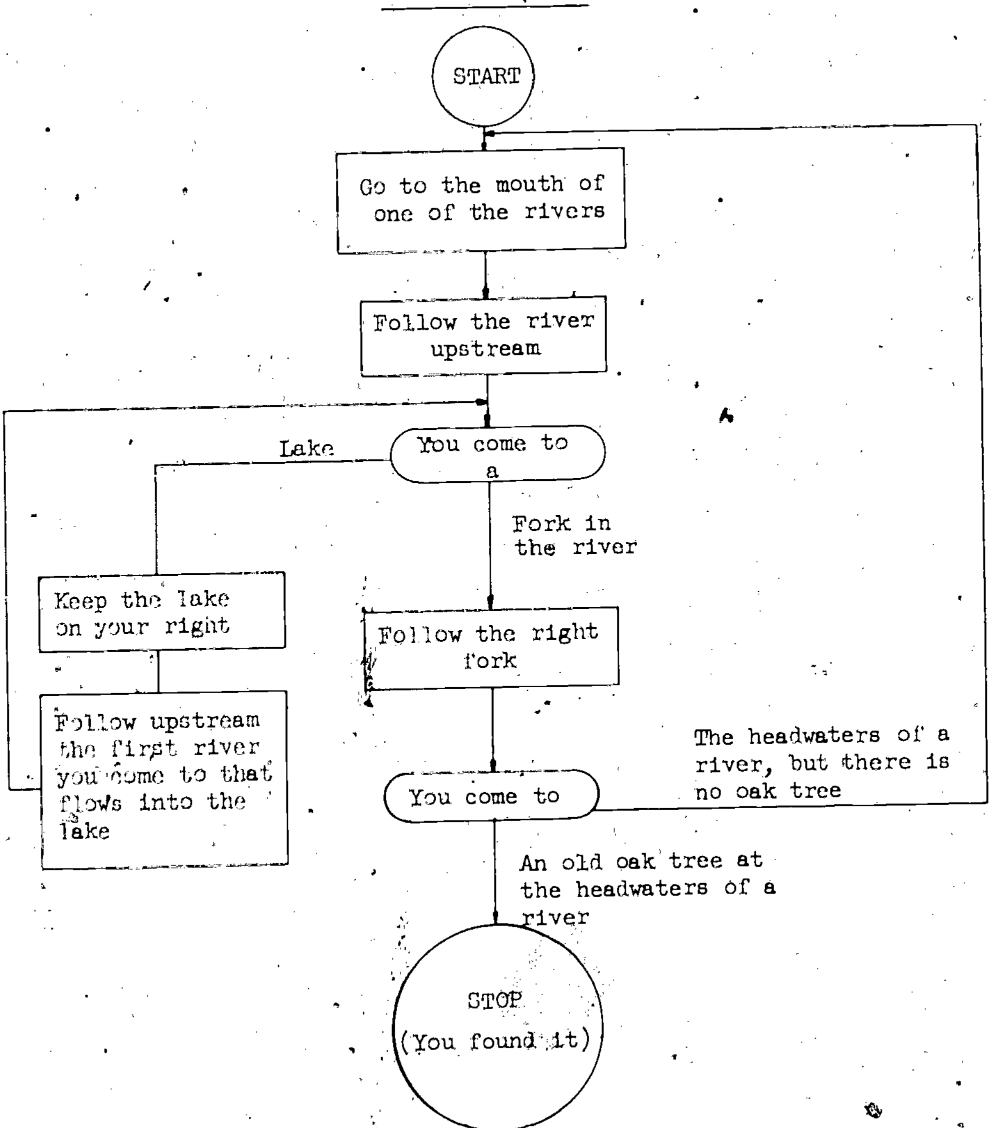


Figure 6

3-3. Assignment and Variables

In computing work a variable is a letter used to represent a number. You have seen examples of letters used in this way in such formulas as

$$A = L \cdot W,$$

where L and W stand for the length and width of a rectangle and A represents its area. As another example we have the formula

$$W = R \cdot T$$

for computing wages. Here, R stands for the hourly rate of pay of a worker in dollars, T for the time worked in hours, and W for his wages in dollars.

In computing work, at any particular time, a variable must represent one definite number. This number is called the value of the variable. Although at any particular time each variable has just one definite value, the value may change from time to time. For example, we might wish to use the formula, $W = R \cdot T$, to compute the wages of several workers who may have different pay rates or may work for different periods of time. Soon we will develop a flow chart for doing just that.

Before drawing this flow chart we will devise a model which will show very clearly how variables are used in computing.

We imagine that for each variable used in our problem there is an associated window box. On top of each box is engraved the associated variable. Inside the box is a slip of paper with the present value (or current value) of the variable written on it. The variable is a name for the number that appears inside.

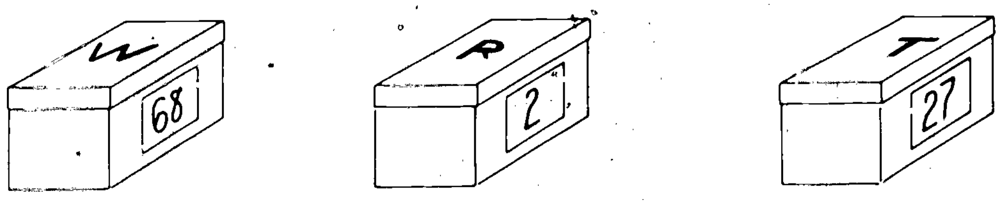


Figure 7. Memory

Each box has a lid which may be opened when we wish to assign a new value to the variable. Each box has a window in the side so that we may read the value of the variable without changing the value. These window boxes make up the memory of our computer.

We imagine that the computing operation is performed by a "master computer" and two assistants called the "assigner" and the "reader". (In a real computer their tasks are performed by electrical circuits.) The master computer receives his instructions from a flow chart and gives certain tasks to the assistants.

Suppose we wish to have a worker's wages computed using the formula

$$W = R \cdot T.$$

The instruction to compute the value of W will come to the master computer in the following flow chart box.

$$\boxed{W \leftarrow R \cdot T}$$

Inside this box we find an assignment statement. To read this statement aloud we say,

"Assign to W the value of $R \cdot T$."

The left pointing arrow is called the assignment operator. This arrow is to be thought of as an order or a command. Rectangular flow chart boxes will always contain assignment statements. Such a rectangular box is therefore called an assignment box.

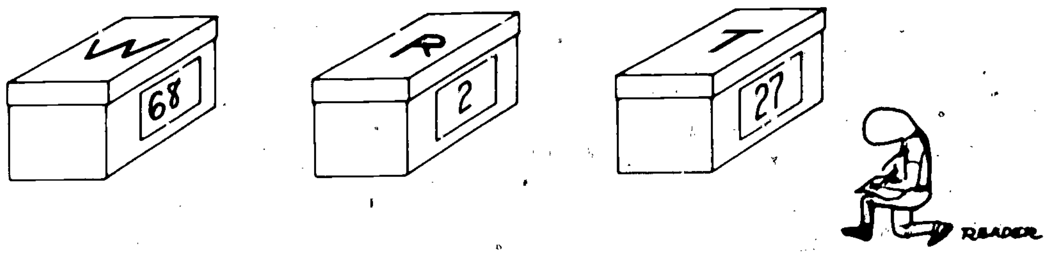
Next, we shall see what happens when the master computer receives such an instruction. We shall assume that R and T (but not W) already have the desired values, say those shown in Figure 7. (How they obtained these values will be discussed later on.)

The computation called for in the assignment statement occurs on the right-hand side of the arrow, so the master computer looks there first.

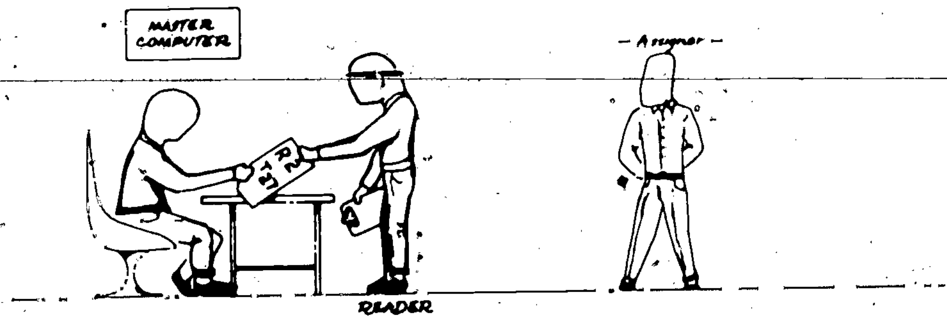
$$\boxed{W \quad \longleftarrow \quad R \cdot T}$$

He sees that he must know the values of R and T. So, he calls the reader and sends him out to bring these values from the memory.

The reader goes to the memory and finds the window boxes labeled R and T. He reads the values of these variables through the windows,



writes the values down, and takes them back to the master computer.



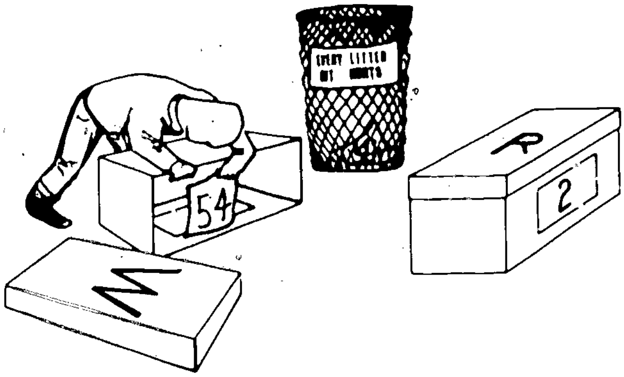
The master computer computes the value of $R \cdot T$ using the values of R and T brought to him by the reader. He gets the value 54 for $R \cdot T$.

Now the master computer looks at the left-hand side of the arrow in his instruction.

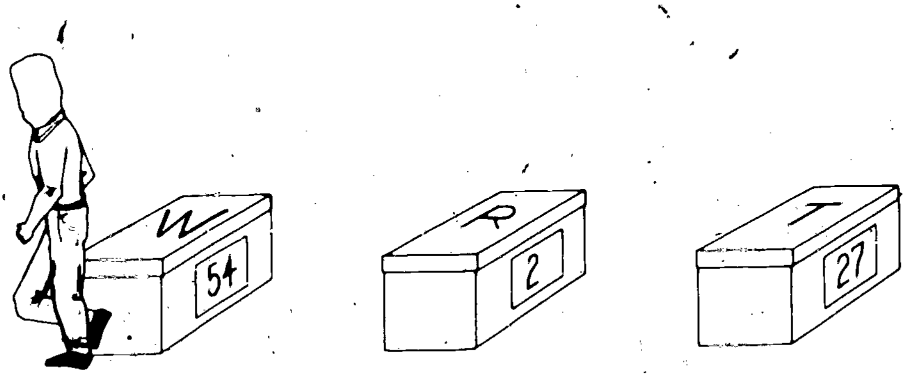
$$\boxed{W \quad \longleftarrow \quad R \cdot T}$$

He sees that he must assign the computed value of $R \cdot T$, namely 54, to the variable W. So he writes "54" on a slip of paper, calls the assigner, and tells him to assign this value to the variable W.

The assigner goes to the memory, finds the window box labeled W, and dumps out its contents.



Then he puts the slip of paper with the new value in the box, closes the lid, and returns to the master computer for a new task.



We say that assignment is <u>destructive</u> because it destroys the former value of the variable. Reading is <u>nondestructive</u> because this process in no way changes the values of any of the variables in the memory.

<u>Check Your Reading</u>

1. In computing work what is a variable?
2. What will you always find inside an <u>assignment box</u>?
3. The left pointing arrow is to be thought of as an ______ or a ______.
4. Why do we say assignment is destructive?
5. Why do we say reading is nondestructive?

Exercises 3-3a

(Class Discussion)

1. (a) The starting (or initial) values of B and C are given in the table below. Fill in the values of these variables after carrying out the instruction in the assignment box on the right.

	B	C
starting values	2	4
final values		

$\boxed{B \leftarrow C}$

(b) Instructions the same as in part (a).

	B	C
starting values	2	4
final values		

$\boxed{C \leftarrow B}$

(c) If we compare the values of B and C after either of the assignment statements

$$B \leftarrow C \qquad \text{or} \qquad C \leftarrow B,$$

what do we find?

(d) Are the effects of the assignment statements

$$B \leftarrow C \qquad \text{and} \qquad C \leftarrow B$$

the same or different? Why?

2. (a) The starting values of A, B, and C are given in the table. The two assignments on the right are to be performed in the indicated order. Fill in the values in the table.

	A	B	C
starting values	9	11	13
values after first assignment			
final values			

$A \leftarrow B$

$B \leftarrow C$

(b) Instructions the same as in part (a).

	A	B	C
starting values	9	11	13
values after first assignment			
final values			

$B \leftarrow C$

$A \leftarrow B$

(c) In what way are the instructions in the assignment boxes of parts (a) and (b) the same?

(d) In what way are they different?

(e) Does the order in which two assignment statements are carried out affect the final result?

Exercises 3-3b

Compute the values of V and A according to the two assignments on the right for each set of values of L, W, and H given in the table below.

	L	W	H	V	A
1.	7	3	2		
2.	8	3.5	5		
3.	11	9	7		
4.	12	2.3	4.6		

$\boxed{V \leftarrow L \cdot W \cdot H}$

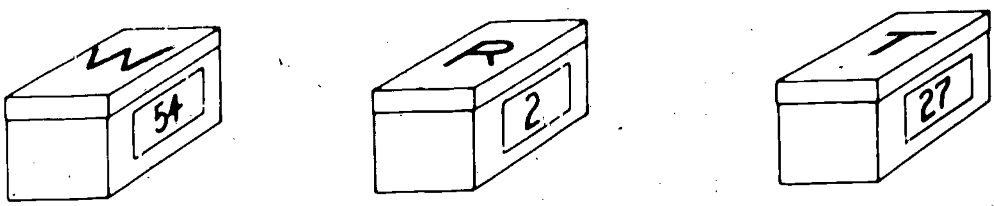
$\boxed{A \leftarrow 2 \cdot [(L \cdot W) + (W \cdot H) + (H \cdot L)]}$

3-4. Input and Output

In the previous section the master computer was instructed to perform the following assignment:

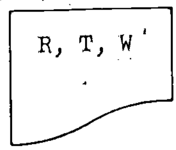
$$\boxed{W \longleftarrow R \cdot T}$$

The values of R and T at the time were 2 and 27. The task was performed with the result that the memory looked like this:



The desired value of W is now stored in the computer's memory. Next we want the computer to produce the answer so we can see it. This will require an instruction to the master computer to print out or output the answer. While we are about it we may as well have the computer print the values of R and T along with the value of W. In this way, in case we have to compute the wages for several workers we will know which wages go with which values of R and T.

Our instruction to the master computer to output the values of R, T, and W takes this form:



We see that we have a new shape of flow chart box. Inside the box, separated by commas, are the variables whose values we wish to know. This box is called an output box. When the master computer receives this instruction he sends the reader to bring to him the values of these variables. When the reader returns with these values the master computer types them out for all to see in the same order as they are listed in the output box.

The shape chosen for our output box suggests a page torn off a line printer, one of the most common computer output devices.

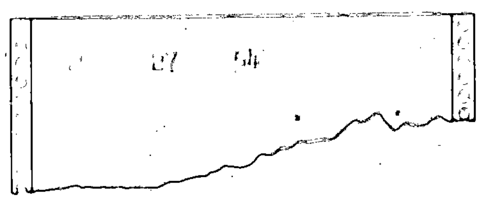
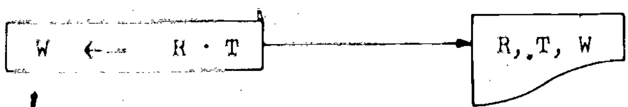


Figure 3. Output from line printer

In Figure 3 we see how the output data for our problem might look if printed by a line printer.

Putting our two flow chart boxes together in the proper order we now have:



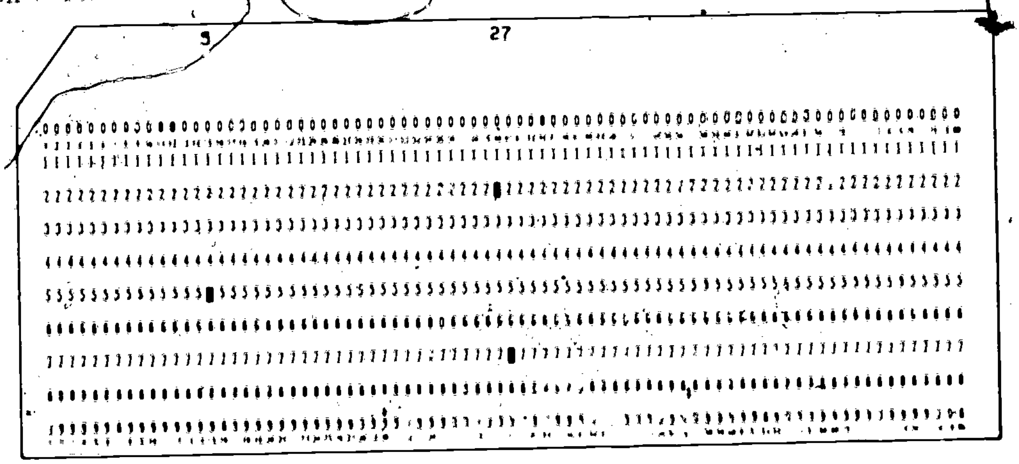
We notice that the actual numbers, the values of the variables being output, do not appear in our flow chart.

Now we give our attention to how the variables R and T get their values. Remember that we are making a flow chart for finding the wages of a worker whose hourly rate of pay and hours worked are given. The rate of pay and hours worked must be read in as <u>input</u> from outside the machine. Just as the actual output values of our variables do not appear in our flow chart, neither do the input values of our variables.

Instead, the master computer will be given an instruction to take whatever values are supplied from outside the machine and assign these values to R and T. This instruction will take the form

R, T

The shape of this [illegible] shown below.



You may have seen such cards and referred to them as "IBM cards". We will call them punch cards or "Hollerith cards" after Herman Hollerith who invented them in the late [illegible] century. The holes punched in the card are a special code for the numbers printed directly above them.

When we see an <u>input box</u> like

R, T

we know there will be a stack of punch cards, each card having two numbers printed on it. When the master computer receives the instruction

R, T he reads the first card in the stack. He then gives the values printed on this card to the assigner to be assigned respectively to R and T. <u>The card which was read is removed from the stack.</u>

Before proceeding to the next part of flow charting, let us <u>review</u> the ideas studied thus far. We have discussed four kinds of flow chart boxes.

The <u>assignment box</u>. This box is rectangular in shape and always contains an arrow pointing left.

$$\boxed{W \leftarrow R \cdot T}$$

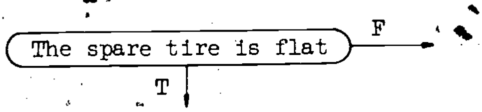
On the left-hand side of this arrow we always find a single variable. On the right-hand side of the arrow we find an arithmetic expression. The assignment box is a <u>command</u> to:

(1) read from the computer's memory (window boxes) the values of any variables occurring to the right of the arrow;

(2) using these values for the variables, compute the value of the expression on the right of the arrow;

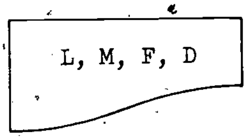
(3) assign this value to the variable on the left of the arrow (that is, put this value in the associated window box).

The <u>decision box</u>. This box is oval in shape and always has two exits, one labeled T (true) and the other labeled F (false).



The decision box always contains a statement instead of an instruction. After checking whether the statement is true or false, we leave the box at the corresponding exit and go on to the next activity.

The <u>output box</u>. This box is shaped like a sheet of paper torn off a line printer.



Inside the box we find a single variable or a list of variables separated by commas. No computation takes place in an output step. The output box is a command to:

(1) read the value of each listed variable from its window box;

(2) print out these values in the order listed.

The input box. This box is shaped like a punch card.

A, Q, B

Inside the box we find a single variable or a list of variables separated by commas. No computation takes place in an input step. The input box is a command to:

(1) read, for each listed variable, a value supplied from outside the computer;

(2) assign these values in order to the variables in the list.

We note that assignment is called for in an input box as well as in an assignment box. The difference is that: in an assignment box, the assigned values are obtained from calculations done inside the computer using values obtained from inside the computer; but in an input box, no computation is involved and the values come from outside the computer.

Exercises 3-4a

(Class Discussion)

1. Which of the following are valid input boxes? If not valid, tell why.

(a) Y

(b) A + 2

(c) 5

(d) A,B,C

(e) A,B,C,D,E,F

(f) H × A

(g) A ⟵ 7

2. Which of the following are valid output boxes? If not valid, tell why.

(a) Z, X

(b) Z + X

(c) Z X

(d) 13

(e) B ⟵ A + 1

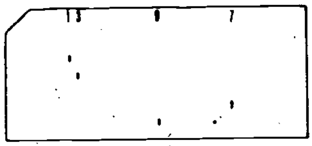
(f) A + 2

(g) P,R,I,N,T

3. (a) What do you think would happen if we had an input box like this:

A, B, A

(b) Suppose that the punch card to be read is:



What values would be in the memory after the card has been read?

4. (a) What do you suppose that the computer would print out for an output box like this:

C, Y, C

(b) Suppose that the computer memory contains the values 19 and 11 for C and Y, respectively. What would the computer print out?

5. Which of the following are valid assignment boxes? If not valid, tell why.

(a)	A, B, C	(e)	2 ← A	(i)	A ← B, C
(b)	A ← B + C × D	(f)	2 ← 1 + 1	(j)	A ← B or C
(c)	A + C → B	(g)	2 ← 2	(k)	A = L × W
(d)	A ← 2	(h)	A + B + C	(ℓ)	V ← E × E × E

Now, let us return to the flow chart for the computation of wages.

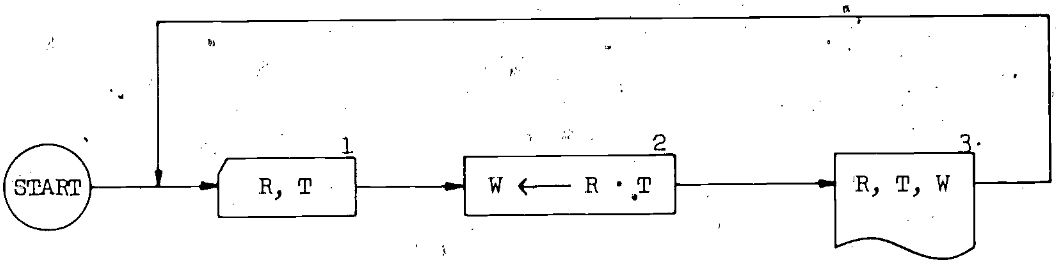
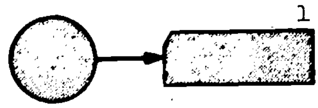
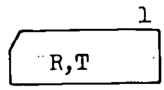
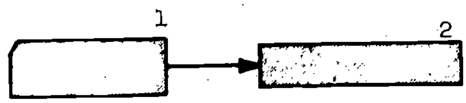
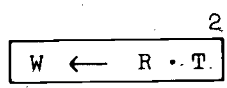
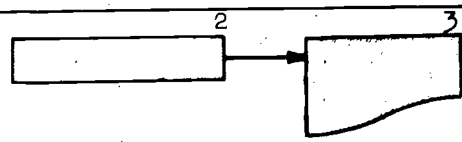
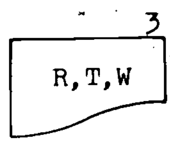
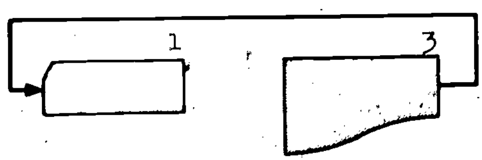
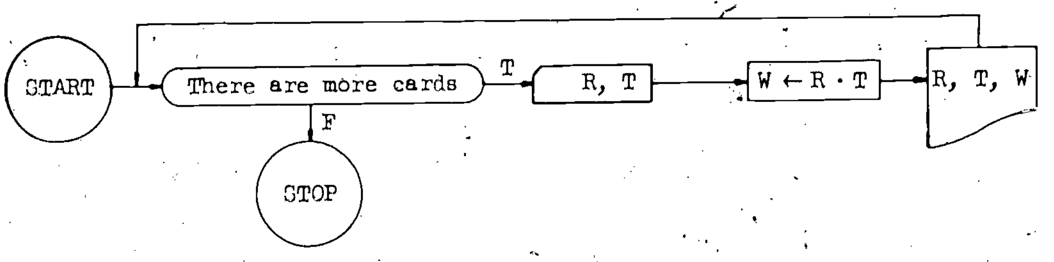


Figure 9. Flow chart for computing wages.

The steps involved in this flow chart, together with the flow chart boxes which call for these actions, are shown on the following page.

STEP	ACTION	CORRESPONDING FLOW CHART BOXES
1	Start.	
2	Go to box 1.	
3	Input two numbers and assign them to R and T.	
4	Go to box 2.	
5	Compute the value of R · T and assign this value to W.	
6	Go to box 3.	
7	Read the values of R, T, and W and output these values.	
8	Return to box 1 and repeat the process with a new set of data values.	

You may have noticed that the flow chart, in Figure 9, provides no instruction to stop. Without such an instruction the flow chart would suggest an "endless loop". We could introduce a decision box before box 1 so that our flow chart would look like this:



We will not ordinarily do this. The reason is that one of the jobs of the input box is to stop the computing process when there are no more cards to be read. Most computers require a special card at the end of a data deck which tells the computer to stop. However, we will agree that

> If a flow chart arrow carries us into an input box and it turns out that there are no cards left in the stack, then the computation is to stop.

To help you really understand assignment and variables we urge you to act out the operations of a simple computer that are described in the following Class Activity Exercises.

Exercises 3-4b

(Class Activity)

A Simple Computer

The Parts: The Master Computer, The Assigner, and The Reader.

Materials: Three window boxes (shoe boxes with holes cut in the side will do), blackboard, chalk, pencils and two pads of paper.

To prepare for the operation of the computer:

(1) Mark the top of one window box with the letter R, another T, and a third W.

(2) Put the following inputs on a deck of six cards and put the output headings on the blackboard.

INPUT			OUTPUT	
R	T	R	T	W
2.00	27			
2.50	38			
1.75	36			
2.10	40			
2.25	3[illegible]			
1.[illegible]	40			

Operating the computer:

(1) Each line in the input list represents a punched card.

(2) The Master Computer should work through the flow chart of Figure 9.

(3) After starting, the Master Computer should read box #1 which tells him to input values of R and T. He tells the Assigner, "Pick up the first data card and put the value of R in the R box and the value of T in the T box."

(4) The Master Computer then reads box #2. He says to the Reader, "Go read the values of R and T, write them down, and bring them back to me."

(5) The Master Computer then computes the value of $R \cdot T$. He then says to the Assigner, "Take this value of $R \cdot T$ and put it in the W box."

(6) The Master Computer reads box #3 next. He says to the Reader, "Go to the R, T, and W boxes, write down these values, and bring them back to me."

(7) The Master Computer then tells the Assigner to write these values on the board under the appropriate Output headings.

(8) Repeat steps (3) through (8) until all data cards are used.

To more closely parallel the operation of a computer the Reader and the Assigner should really perform only one task at a time. One can observe that "reading the value of the variable" is nondestructive and that "assignment" is destructive.

Also as the "simple computer" operates it is helpful to observe the role of the variable in computing work. One can see that the variable represents, at any particular time, one definite number, and that the value of the variable may change from time to time.

Exercises 3-4c

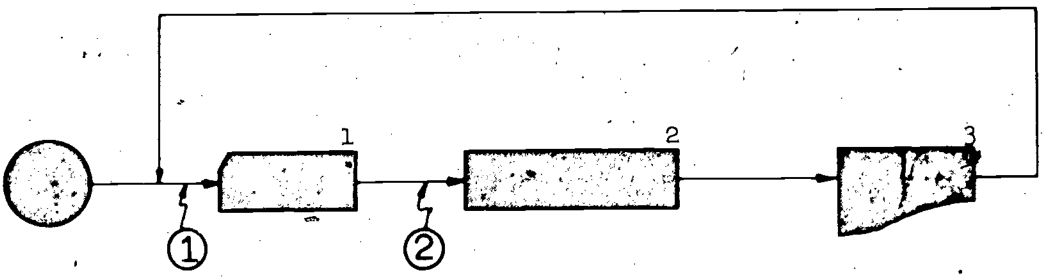
1. The flow chart of Figure 9 is to be carried out with four sets of values of R and T (four punch cards). The values on these cards are found in the table below:

	R	T
First card	2	27
Second card	2.15	39
Third card	1.87	41.75
Fourth card	1.945	37.25

(a) Display the output using one card for each time through the output box 3.

(b) Same as part (a) but this time round off the wages to the nearest penny. (R is given in dollars per hour and T in hours.)

2. The flow chart of Figure 9 is shown in silhouette.



Using the data shown in problem 1, find the values of the variables R, T, and W at each of the following stages:

(a) the second time we arrive at the point marked ①,

(b) the third time we arrive at ②,

(c) the last time we arrive at ②,

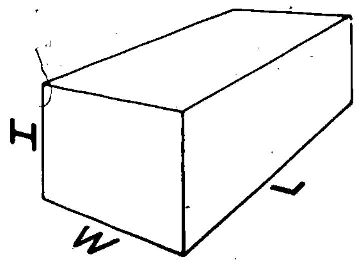
(d) the first time we arrive at ②,

(e) the first time we arrive at ①.

[Note: In some parts of this question you will be unable to give the values of some of the variables. When this happens indicate which variables do not have their values determined by the available information.]

3. The volume, V, of a box is given by the formula

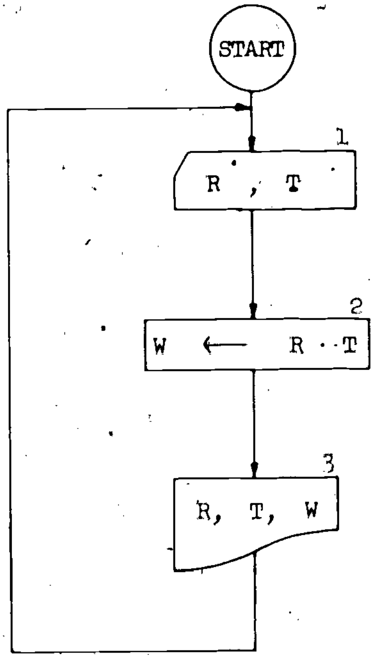
$V = L \cdot W \cdot H.$



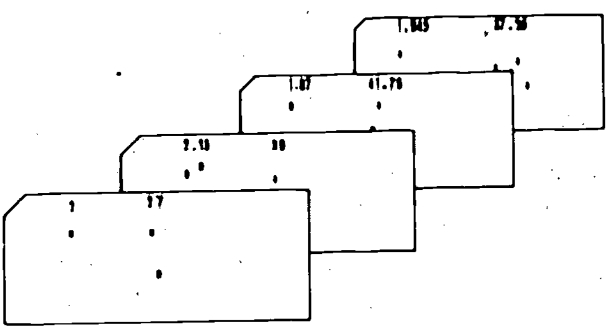
Draw a flow chart for inputting various values of L, W, and H, computing the values of V, and outputting the computed values.

3-5. Using a Variable as a Counter

In the last section we built a flow chart for computing the wages of employees. If we draw this flow chart vertically instead of horizontally it looks like this:



If the input data are the following:



then the output values look like this:

2	27	54
2.15	39	83.85
1.87	41.75	78.0725
1.945	37.25	72.45125

We might wish to have our lines of output numbered for easy reference so as to appear as follows:

1	2	27	54
2	2.15	39	83.85
3	1.87	41.75	78.0725
4	1.945	37.25	72.45125

In order to number the lines of output we put an extra variable, which we will call N, in our output box.

N, R, T, W

We would like to make the variable N take on the values 1, 2, 3, ..., in order. To do this numbering we place in our flow chart an additional assignment box.

N ⟵ N + 1

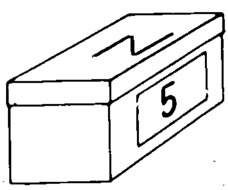
To see what this instruction means we remember that an assignment statement is a command to:

first, look up the values of the variables on the right;

second, using these values compute the value of the expression on the right;

third, assign this computed value to the variable on the left.

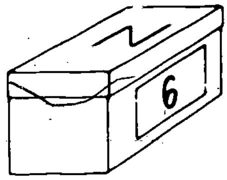
To see how this works out with the instruction, N ← N + 1, suppose that the value of N is 5 before carrying out the instruction. We look up the value of N,



Window box before

30

which is 5; we compute N + 1, which is 6; we assign this value to N.



Window box after

The effect of this instruction, then, is to increase the value of N by 1. This is just what we wanted. So we place this box in our flow chart as follows:

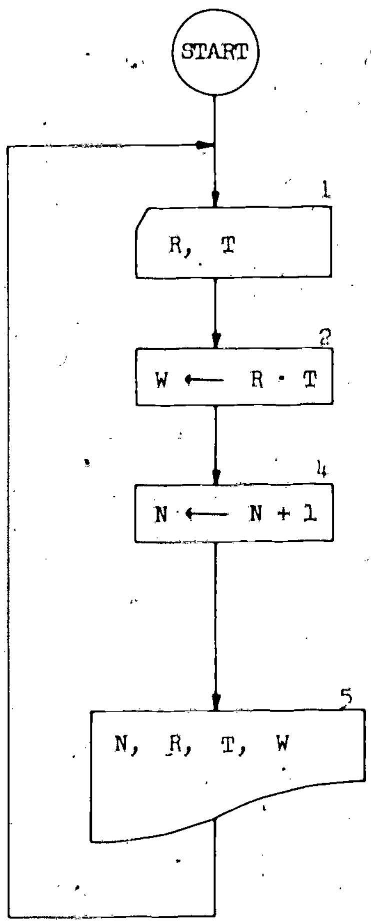
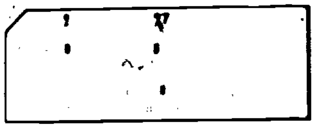


Figure 10

Exercises 3-5a

(Class Discussion)

1. Trace the pair of data values



   through the flow chart in Figure 10.

2. What is your instruction when you come to the newly added assignment box $\boxed{N \leftarrow N + 1}$?

3. Is it possible to follow this instruction?

4. What change in the flow chart must be made so that the instruction $\boxed{N \leftarrow N + 1}$ can be followed?

To solve this problem we give the variable "N" an "initial" or starting value. This must be done just once. Therefore, we put the instruction in an assignment box outside the loop, as shown in Figure 11a.

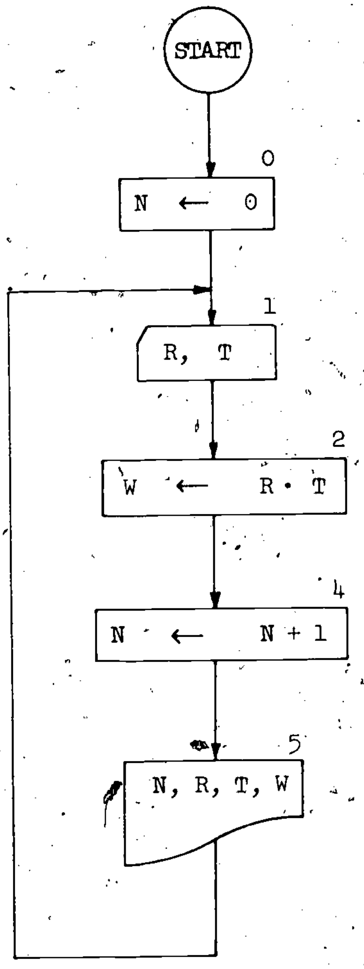


Figure 11a

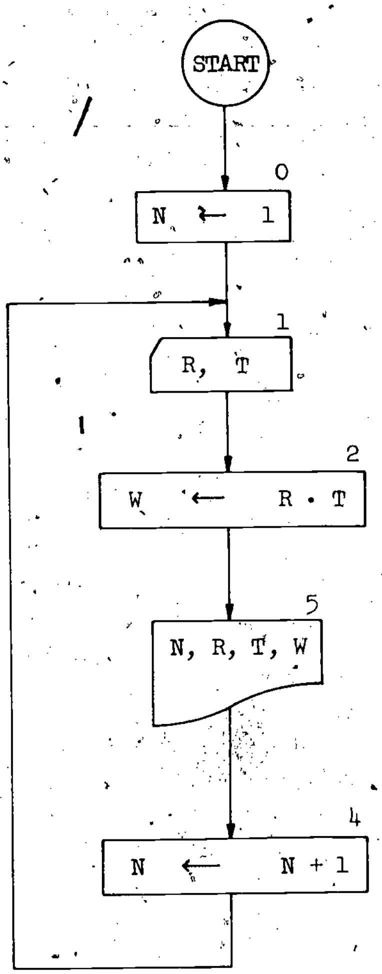


Figure 11b

In Figure 11a, we assigned the variable N a starting value of 0 instead of 1. If we begin with the value 1, it would be stepped up to 2 in box 4 before any output. The first line of output would then be numbered 2, a result which of course we do not want.

It is possible to start N off with the value 1 if we rearrange the boxes. The flow chart in Figure 11b achieves the same result as that in Figure 11a. In Figure 11b we step up the value of N after, rather than before, the output step. This probably seems more natural. However, Figure 11a has the advantage that we can simplify it in the following manner:

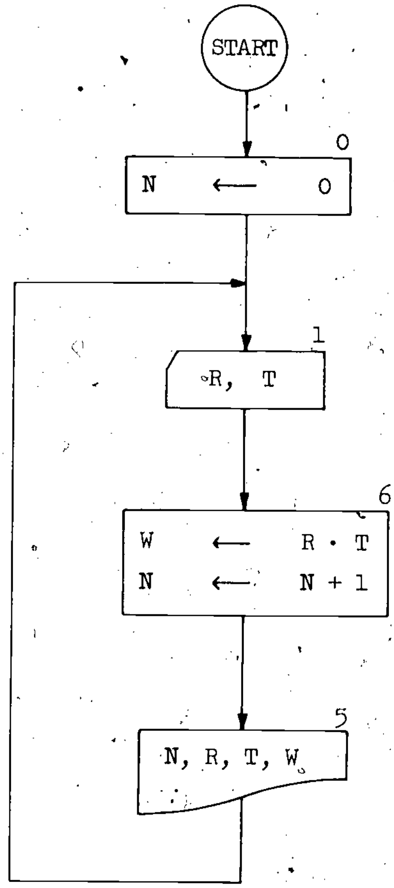


Figure 12

Whenever two or more assignments are called for in succession with no other steps in between, we will allow all the assignment steps to be put into one flow chart box with the understanding that <u>these steps are to be carried out in order reading from top to bottom</u>.

Exercises 3-5b

(Class Discussion)

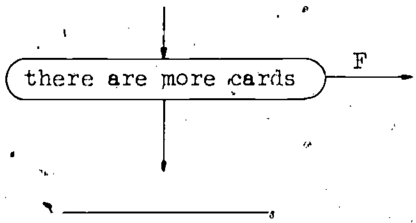
1. An employer using the flow chart of Figure 12 would also like to know the total amount of his payroll (that is, the total of all the wages paid). This can be accomplished by introducing a variable P (for payroll) into our flow chart. Each time a worker's wages are computed, the value of P is increased by the value of W.

   (a) Write the assignment statement that orders the Master Computer to increase the value of P by the value of W.

   (b) Write the assignment statement that assigns the starting value of P.

   (c) When will P have the desired value (that is, the sum of all the wages paid)?

   (d) Revise the flow chart of Figure 12 to include the above features and to provide for the output of only the final value of P. Hint: you will want to use the flow chart box:



Exercises 3-5c

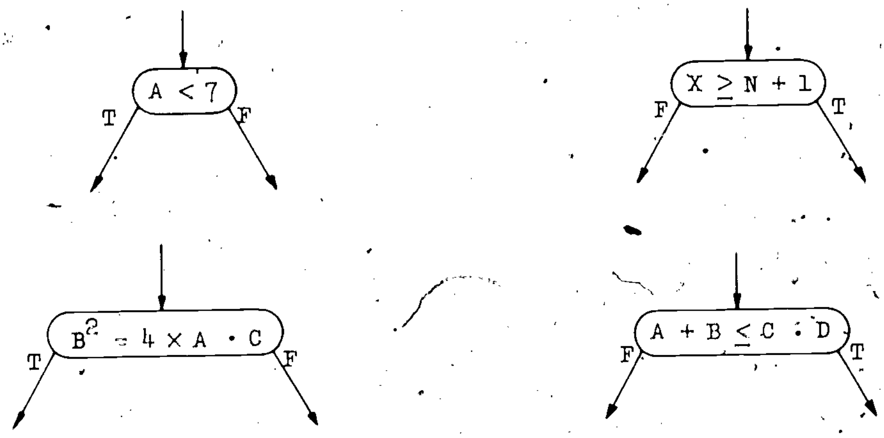
1. Using the flow chart developed in Class Discussion Exercises 3-5b, write the complete output for each of the following pairs of input data:

	R	T
(a)	2.50	32
(b)	3.00	38
(c)	3.40	22
(d)	2.75	40
(e)	3.60	39

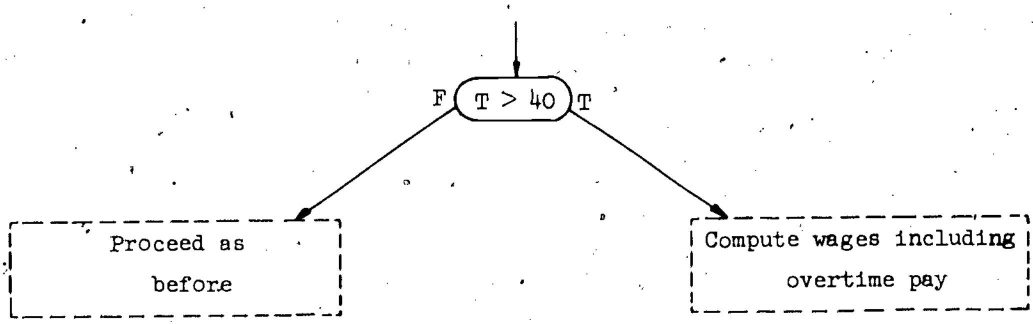
3-6. Decision and Branching

In our informal "flat tire" and "lost in the woods" flow charts we have seen decision boxes. Each decision box with its two exits introduces a branching. Often one or both branches lead into a loop. We have already noted the importance of looping for computers.

We are ready to consider mathematical flow charts involving decision boxes. The statements that appear inside these decision boxes are mathematical statements, either equations or inequalities. Some examples are:



As a simple example of the use of a decision box, suppose that we wish to include an overtime feature in our payroll flow chart. If an employee receives double pay for all hours worked over forty, then we need the following decision box in our flow chart:



Before proceeding with the development of our flow chart we need to find a formula for the wages of an employee which will include the pay he receives for working overtime. The following exercises will develop such a formula starting with the familiar formula

$$W = R \cdot T$$

where W represents his wages, R his pay per hour, and T the number of hours worked.

Exercises 3-6a

(Class Discussion)

1. If the employee works 40 hours for R dollars an hour, then write an expression that represents his <u>wages</u> for the 40 hours.

2. The employee is paid <u>double</u> the hourly rate when he works overtime. Write an expression representing his rate of pay per hour for overtime where R is his regular hourly rate.

3. If T represents the total number of hours worked, then write an expression which represents the number of hours of overtime (i.e., the number of hours he works in excess of 40 hours).

4. Having found the rate of pay for overtime, (Exercise 2), and the number of hours of overtime, (Exercise 3), write an expression representing the wages for overtime work.

5. To the regular wages of the employee we must add his wages for overtime work. Write an expression which represents the total wages of the employee that includes the pay he receives for working overtime.

By using the assignment box

$$\boxed{W \leftarrow R \cdot 40 + 2 \cdot R \cdot (T - 40)}$$

we obtain the following flow chart which provides for extra pay for overtime, for output of the weekly wages of any employee, and for the total amount in the whole payroll.

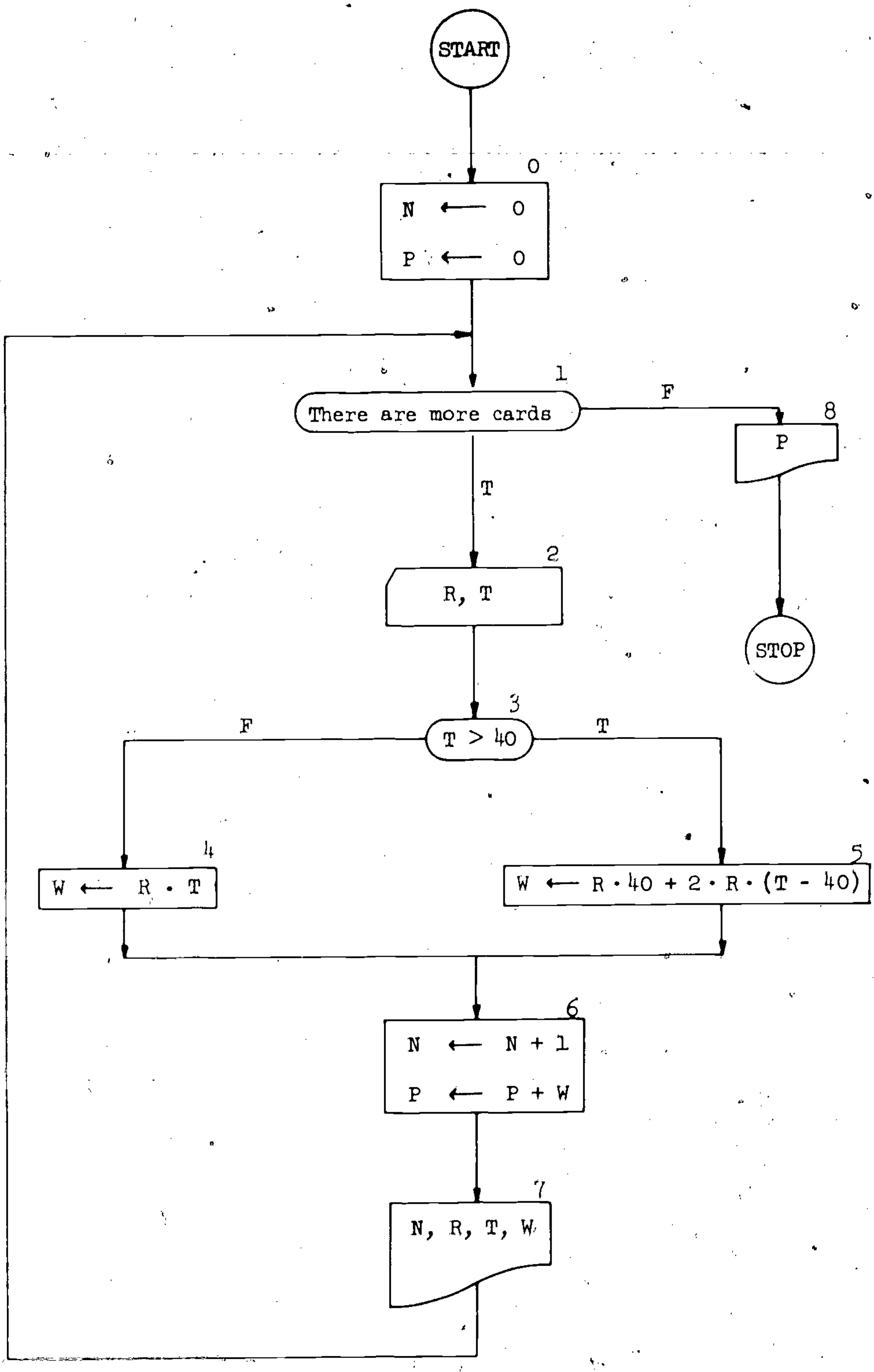


Figure 13. Flow chart for payroll, including double rate for overtime.

Notice that there are two output boxes in Figure 13, one for the wages of the individual employee and one for the total payroll.

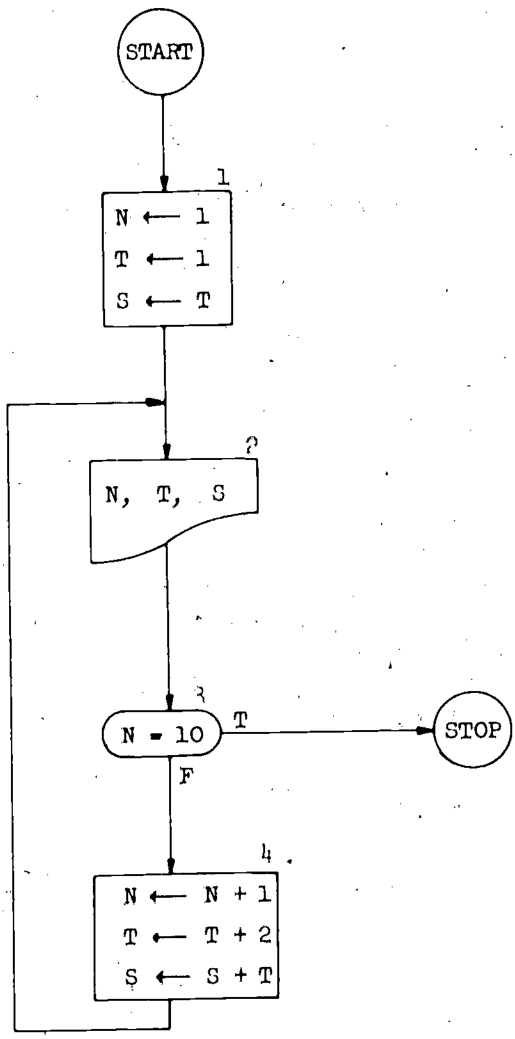
Exercises 3-6b

1. Trace through the flow chart of Figure 13 with the input given below. Give the output for each time through the output box 7, and finally the output for the total payroll (P), box 8.

	R	T
First card	2.15	39
Second card	2.64	44
Third card	1.98	27
Fourth card	2.15	40
Fifth card	2.26	45

2. If two assignment statements occur in the same assignment box, give conditions under which the two statements may be interchanged without changing the values which will be assigned to any variables.

3. Trace through the following flow chart and give the output values. In carrying out such a trace you should have a piece of paper on which to list the output values and a scratch pad on which you keep a running record of the latest value assigned to each variable. Each time you assign a new value to a variable, cross out the old value and write down the new one. The appearance of the output sheet and the scratch pad are given below for the first three times through the loop.



OUTPUT

N	T	S
1	1	1
2	3	4
3	5	9
4	7	16

SCRATCH PAD

N	~~1~~	~~2~~	~~3~~	4
T	~~1~~	~~3~~	~~5~~	7
S	~~1~~	~~4~~	~~9~~	16

4. Answer these questions about Exercise 3.

(a) Describe in words the output list of values of N.

(b) Do the same for the list of output values of T.

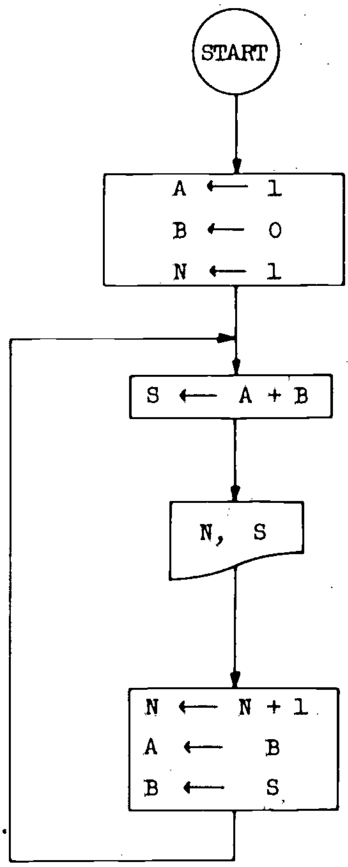
(c) Each value in the list of values of S (after the first one) can be found by adding what other two numbers in the output list?

(d) What instruction in the flow chart illustrates your answer to part (c)?

(e) Can you express the output values of S entirely in terms of the various output values of T?

(f) Fill in the blanks. The result of part (e) can be expressed by saying that the purpose of the variable S is to keep a running ________ of the values of ________.

5. Trace through the accompanying flow chart and give the output. Carry your work to the stage where N has the value 15.



6. Recall that 2^5 represents $2 \times 2 \times 2 \times 2 \times 2$ and is equal to 32. Similarly, 2^K, K a counting number, represents the product

$$2 \times 2 \times 2 \times \ldots \times 2$$

where there are K factors all equal to 2. This number is called the <u>Kth</u> power of 2. We want to make a flow chart to output each power of two from the <u>first</u> through the <u>20th</u>. We cannot use an instruction with a string of dots in it and we will not permit the use of exponents. The table below will help in figuring out some correct instructions. (P represents the value of the <u>Kth</u> power of 2.)

K	1	2	3	4	5	6			
P	2	4	8	16	32	64			

(a) How is each value of K obtained from the preceding one?

(b) How is each value of P obtained from the preceding one?

(c) Fill in at least five more columns in the table.

(d) Now make your flow chart. Be sure to give K and P starting values and also provide for a stopping mechanism.

7. This problem is similar to problem 6. The number "five factorial" is written 5!, and means $1 \times 2 \times 3 \times 4 \times 5$ which is equal to 120. Similarly, if K is a counting number, then K! is defined as

$$1 \times 2 \times 3 \times 4 \times \ldots \times K,$$

that is, the product of all counting numbers from 1 through K. As in problem 6 we tabulate a few values here. (F represents the value of K!)

K	1	2	3	4	5		
F	1	2	6	24	120		

(a) How is each value of K obtained from the preceding value?

(b) How is each value of F obtained from the preceding value of F and the current value of K?

(c) Fill in two more columns in the table.

(d) Draw your flow chart. Arrange to stop when 15 values are printed out.

Exercises 3-6c

(Class Discussion)

A teacher assigned her students a problem of constructing a flow chart as follows:

(1) The <u>input</u> consists of the lengths and widths of several rectangles.

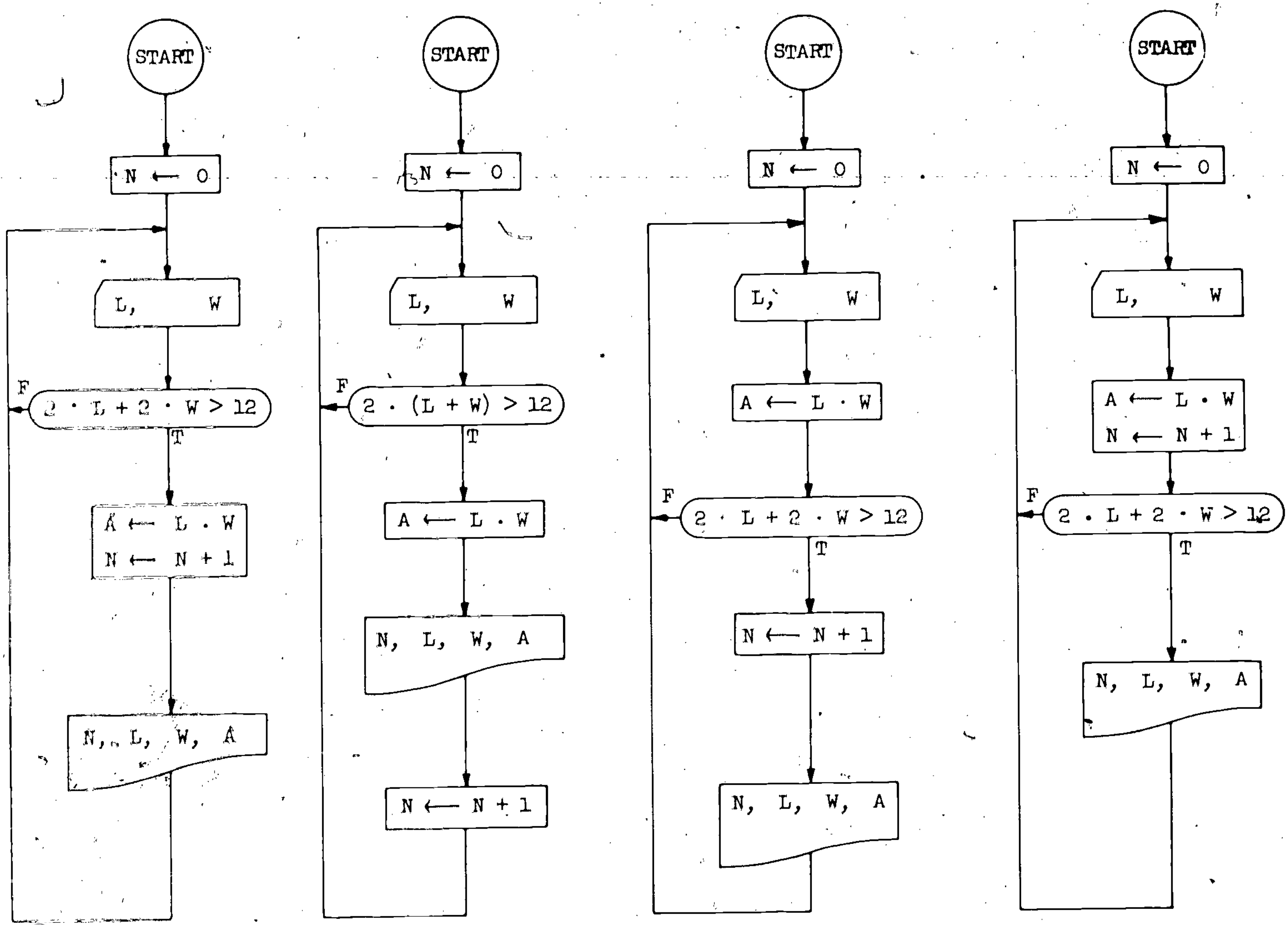
(2) The purpose is to <u>output</u> a list of <u>consecutively</u> <u>numbered</u> lines, starting at one, giving the length, the width, and the area of only those rectangles with perimeter greater than 12.

The flow charts shown on the next two pages were submitted by students as solutions of the problem.

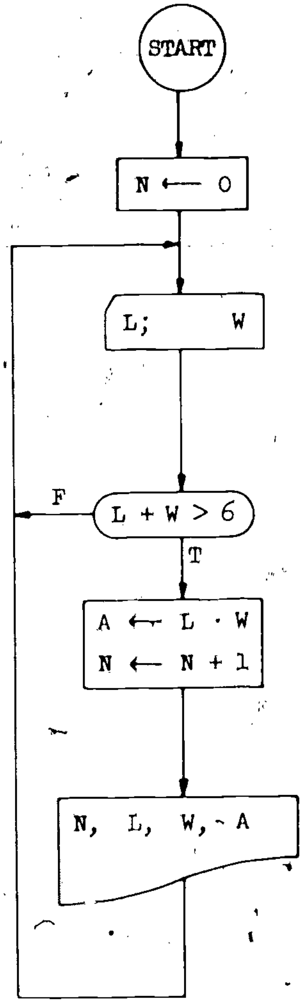
1. Which of the solutions are correct, and which are incorrect?

2. For those that are incorrect, in what way will the answers produced be wrong?

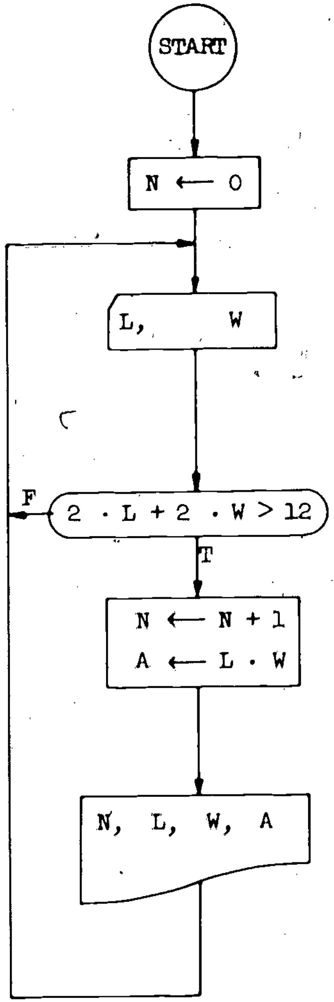
3. For those which are correct, list them in order of efficiency with the one requiring the least amount of computation first.



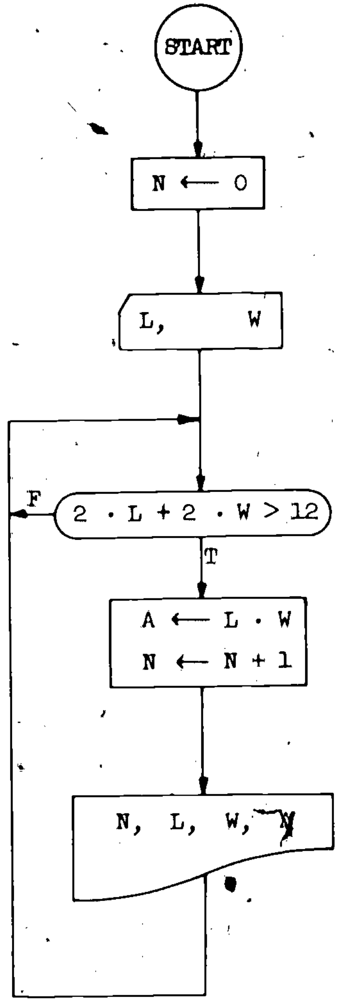
JOHN PAUL GEORGE PETE



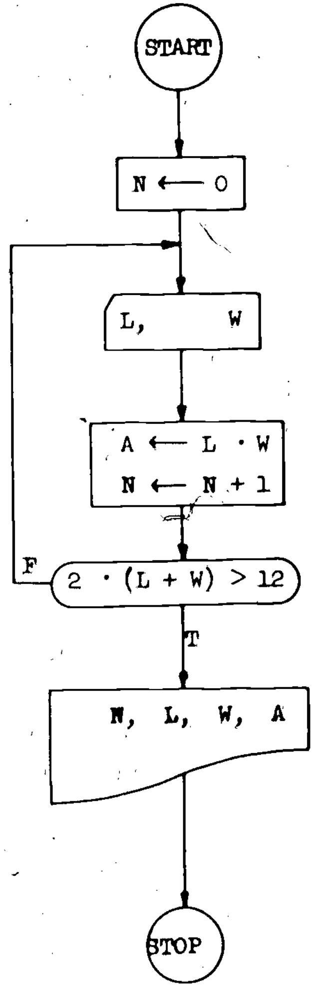
TOM



GORDY



LARS



BOB

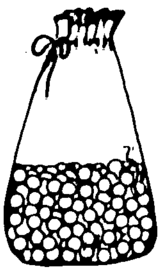
In the previous exercises you have seen that there are several correct flow charts for the desired algorithm. Usually we want to find the most efficient one from a computational viewpoint, but sometimes we want another feature included which may not lead to the least number of calculations. The basic requirement for any flow chart is still, "Will it work?" Simplifying and "streamlining" the flow chart can be accomplished as needed.

3-7. Flow Charting the Division Algorithm

A playground director found a sack of marbles while cleaning up the storeroom. Instead of throwing them away, he decided to divide them among the seven boys on the playground who were helping him. If, after dividing the marbles equally among the boys, there were any left over, he would put the extras away for the time being.

Here is the way the director distributed the marbles. First he lined the boys up.

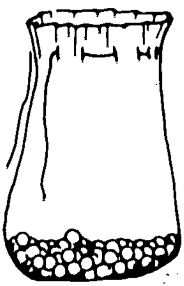
John
Paul
George
Pete
Tom
Gordy
Lars



Then he reached into the sack and took out seven marbles and put one marble in front of each boy. He repeated this process over and over.

Let us take a look at this process somewhere in the middle. We see that the marbles distributed form a rectangular array.

John	O	O	O	O	O	O
Paul	O	O	O	O	O	O
George	O	O	O	O	O	O
Pete	O	O	O	O	O	O
Tom	O	O	O	O	O	O
Gordy	O	O	O	O	O	O
Lars	O	O	O	O	O	O



We know that the total number of marbles in the array is equal to the number of rows times the number of columns (rows are horizontal; columns are vertical). As there are seven boys there are seven rows. Let Q be a variable representing the number of columns. Then the number of marbles distributed so far is

$$7 \cdot Q.$$

We see that the number of marbles already distributed plus those remaining in the sack is equal to the total number of marbles that the director found in the storeroom. Thus, if we let R be a variable representing the number of marbles remaining in the sack and let M represent the number of marbles he had at the beginning, we have the formula:

$$\underbrace{M}_{\substack{\text{Total}\\ \text{number of}\\ \text{marbles}}} = \underbrace{7 \cdot Q}_{\substack{\text{number}\\ \text{distributed}}} + \underbrace{R}_{\substack{\text{number}\\ \text{remaining}\\ \text{in sack}}}$$

This formula is true at every stage of the distribution process.

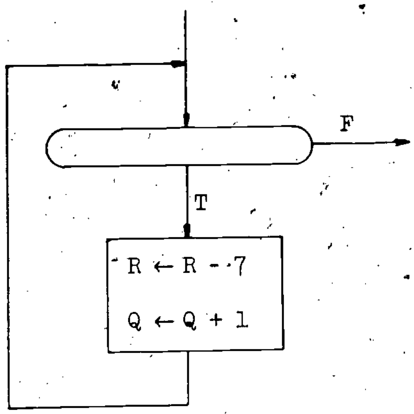
We can recognize the director's process as an algorithm and we will draw a flow chart for it. The basic step is taking seven marbles out of the sack and using them to form a new column of the array. In doing this we decrease R (the number in the sack) by 7 and increase Q (the number of columns) by 1. These activities are represented by the assignment statements:

$$\boxed{\begin{array}{l} R \leftarrow R - 7 \\ Q \leftarrow Q + 1 \end{array}}$$

Since this process is to be repeated over and over we write:

$$\boxed{\begin{array}{l} R \leftarrow R - 7 \\ Q \leftarrow Q + 1 \end{array}}$$

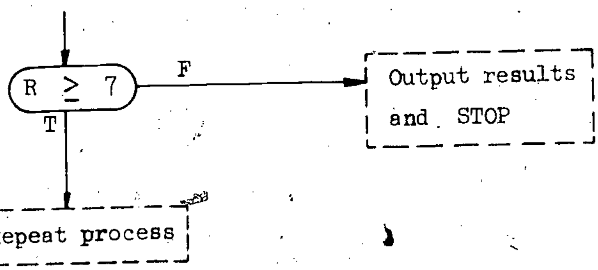
The difficulty with such a loop is that no way is provided for stopping. As you remember, in order to escape from this endless loop a decision box must be introduced.



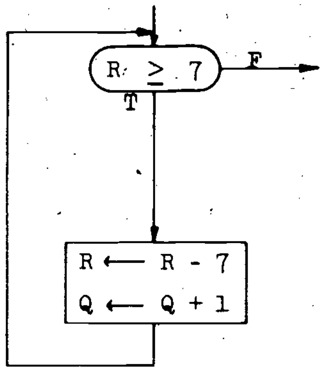
The statement to be placed in the decision box will stem from the fact that in distributing the marbles there comes a time when we cannot remove seven marbles from the sack because there will not be seven marbles left. In other words, in order to remove seven marbles it is necessary that the remainder (of marbles) be greater than or equal to seven, that is,

$$R \geq 7.$$

If we place this statement in the decision box we will provide a means for stopping the process.

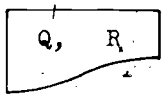


Combining this with the above assignment box we have:

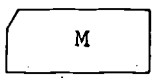


That is the heart of our flow chart. Only minor details remain: namely, to provide output and to give starting values to our variables.

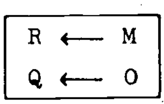
As output we merely give the values of Q and R, thus:



We want to input the value of M:



We start R and Q out with the values they should have before any marbles have been distributed. These are given by:



Putting all our flow chart fragments together we get the complete flow chart.

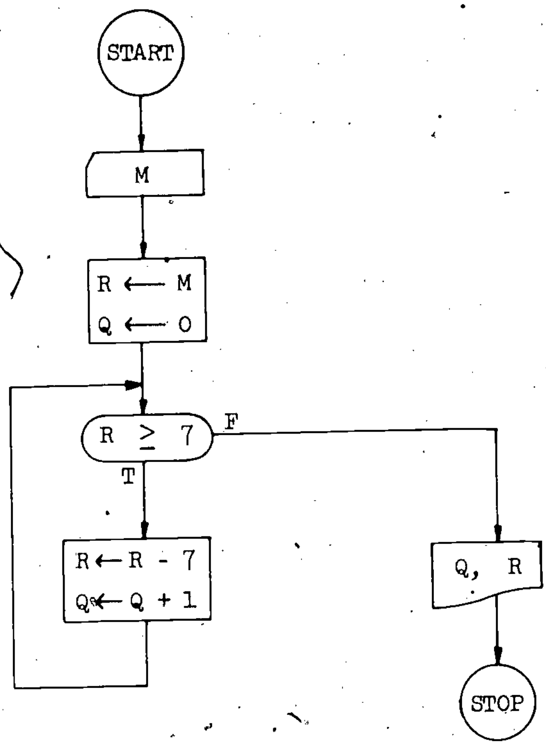


Figure 14

The final value of Q, the value that is output, is the number of columns in our final array of marbles--the number of marbles each boy gets. The final value of R is the number of extra marbles kept by the director and is one of the numbers

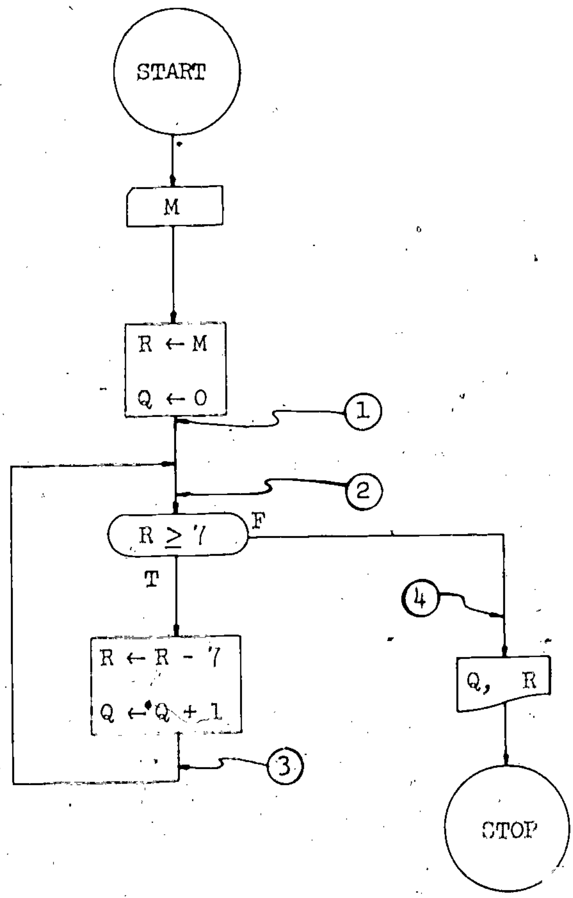
0, 1, 2, 3, 4, 5, 6.

The algorithm can be called the algorithm for <u>integer</u> <u>division</u> with a divisor of 7. We have divided M marbles into 7 piles of Q marbles each with R marbles left over. This is the best distribution that can be made without breaking any marbles.

Exercises 3-7a

(Class Discussion)

1. The flow chart of Figure 14 is shown below.



Using an input value of 17, find the values of the variables Q and R at each of the following stages:

(a) The first time we arrive at the point marked (1).

(b) The second time we arrive at (2).

(c) The second time we arrive at (3).

(d) The third time we arrive at (2).

(e) The first time we arrive at (4).

We see that there are four numbers involved in this integer division process.

$$\underbrace{M}_{\text{Dividend}} = \underbrace{7}_{\text{Divisor}} \cdot \underbrace{Q}_{\text{Quotient}} + \underbrace{R}_{\text{Remainder}}$$

Dividend:	Divisor:	Quotient:	Remainder:
number of things to be divided.	number of piles into which the dividend is to be divided.	number of things in each pile.	number of things remaining undistributed (remainder is less than divisor).

Of course, the same kind of reasoning would work for any divisor. The divisor does not have to be 7. To make a flow chart for integer division for any divisor, we use a variable, D, to denote the divisor. We call for both the dividend and the divisor to be input, and we replace each 7 in the preceding flow chart by D. Then we will have the flow chart:

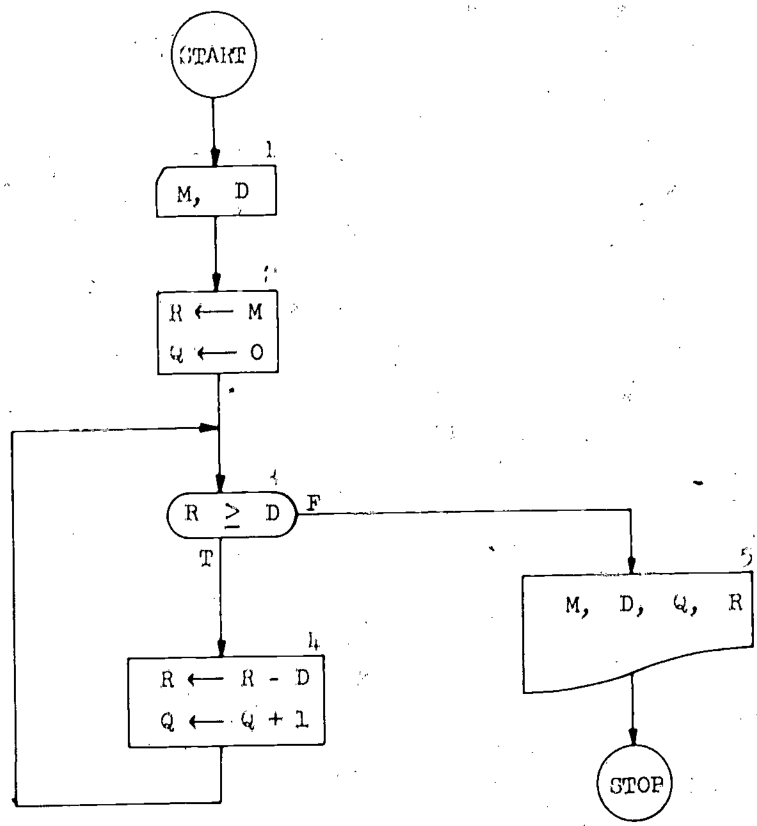
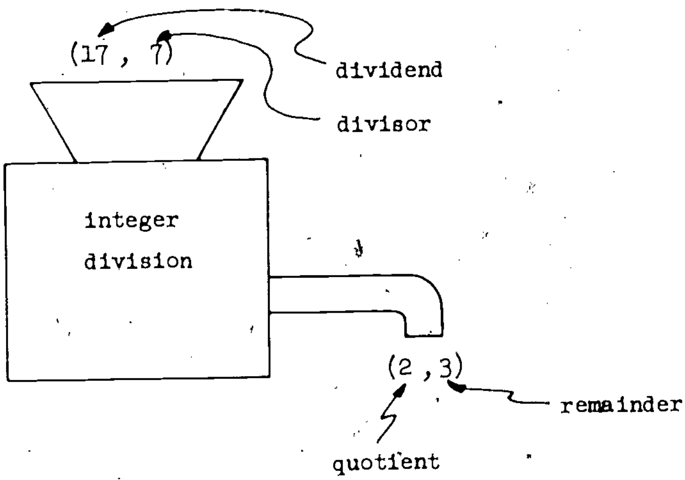


Figure 15. Integer division

This time we have called for the output of the values of all the variables M, D, Q, and R in this order, so as to suggest the formula

$$\underbrace{M}_{\text{Dividend}} = \underbrace{D}_{\text{Divisor}} \cdot \underbrace{Q}_{\text{Quotient}} + \underbrace{R}_{\text{Remainder}}.$$

We can think of integer division as a function whose input is a pair of numbers (the dividend and the divisor) and whose output is another pair of numbers (the quotient and the remainder).



The dividend and the divisor are a pair of whole numbers (with the divisor not equal to 0). The quotient and the remainder form another pair of whole numbers--the only pair of whole numbers Q and R satisfying the two conditions

(a) $M = D \cdot Q + R,$

(b) $R < D.$

During an application of the algorithm the first of these conditions holds at every stage. We repeatedly decrease the value of R by subtracting D until the second condition is satisfied.

Exercises 3-7b

1. Tell what will happen in our algorithm for integer division if we disobey instructions and input the value 0 for D.

2. Trace through the flow chart of Figure 15 with each of the following pairs of input values. For each pair of input values make a table showing the values of all four variables at each passage through box 3 of the flow chart. The last line in the table will be the output of the flow chart. In each of these pairs the dividend is given first and the divisor second.

   (a) (23 , 7)   (d) (0 , 23)
   (b) (24 , 5)   (e) (24 , 8)
   (c) (5 , 24)   (f) (73 , 6)

   We give the solution to part (a) as an example:

	M	D	Q	R
1st	23	7	0	23
2nd	23	7	1	16
3rd	23	7	2	9
4th	23	7	3	2

3. In each of the parts of the preceding problem you note that the values of M and D never change. What is it about the flow chart which explains this?

4. Given the following pairs of values for M and D perform the necessary division and find the corresponding pairs of output values for Q and R.

   (a) (17 , 7)   (f) (17 , 35)   (k) (14 , 1)
   (b) (39 , 13)   (g) (38 , 12)   (ℓ) (62 , 1)
   (c) (47 , 2)   (h) (125 , 125)   (m) (272 , 16)
   (d) (54 , 9)   (i) (194 , 64)   (n) (3168 , 7)
   (e) (53 , 53)   (j) (84 , 12)   (o) (958 , 236)

5. (a) What will be the output values of Q and R if M and D have the same value?

(b) What will be the output values of Q and R if the input value of D is 1?

(c) If you are given the output values of Q and R, are you able to determine what the input values of M and D were? Explain.

3-8. Summary

Section 3-1.

An algorithm is a list of instructions for carrying out some process in a step-by-step, sequential manner.

A flow chart is a diagram which represents the steps in an algorithm.

In a flow chart, commands to take some action are enclosed in rectangular boxes.

In a flow chart, statements on which we are asked to make some decision are enclosed in oval frames, and these boxes always have two exits.

In a flow chart, a loop is a convenient method of handling a repetitive process, but there must be some way out of the loop to prevent it from becoming an endless process.

Section 3-2.

When preparing to solve a problem with a computer it is usually necessary to construct an algorithm for the problem. In preparing the algorithm it is not necessary to know exactly what the computer will do at each step, but it is necessary to provide instructions which, if followed, will lead to the correct answer. In other words, in constructing a flow chart we try to provide for all of the alternative paths that the computer might take even though we do not know exactly what paths our computer will follow at any given stage.

Section 3-3.

In any computing problem, there corresponds to each variable used in that problem a location in the computer's memory. By assigning a number to a variable we mean simply putting the number

destructively into the storage location corresponding to that variable. In evaluating arithmetic expressions a variable is to be treated as a name for the number in the corresponding memory location. The number in the corresponding memory location is referred to as the value (or current value) of the variable. During the course of computation many different values (perhaps even millions) may be assigned to a given variable. Thus it will not be meaningful to speak of the value of a variable without specifying the time, or more precisely, the stage of the computing process. But once the stage of the process is specified, the value of the variable is uniquely determined.

An assignment statement is always placed in a rectangular box like the following: $\boxed{I \leftarrow P \cdot R \cdot T}$. We read this statement

"assign to I the value of $P \cdot R \cdot T$".

This flow chart box is called an assignment box.

Assignment is destructive in that it destroys the former value of the variable. Reading the value of the variable is not destructive.

Section 3-4.

An output box, in a flow chart, contains a single variable or a list of variables which is a command to read the values of the variables and print out these values in the order listed.

An input box, in a flow chart, contains a single variable or a list of variables and is a command to assign the value of the variables, in the given order, to the appropriate place in the memory. Another function of the input box is to stop the computer when all of the data cards have been used.

Section 3-5.

Sometimes we wish to count the number of sets of data assigned to the memory or number the acceptable outputs. We use a variable in an assignment box like the following to accomplish this process.

$$\boxed{N \leftarrow N + 1}$$

You must be sure to assign the variable an initial value such as $\boxed{N \leftarrow 1}$ or $\boxed{N \leftarrow 0}$ so that the counting can begin.

Section 3-6.

Branching is indicated in flow charts by a decision box that is oval in shape. The decision box gives us the ability to choose a new path depending on whether a certain condition is satisfied. One of the functions of the decision box is that it enables us to get out of an endless loop in a flow chart.

Section 3-7.

In flow charting the division algorithm for integer division we think of it as a function whose input is a pair of numbers (the dividend and the divisor) and whose output is another pair of numbers (the quotient and the remainder).

When we input a value of 0 for D in our flow chart (that is, try to divide by zero) we get locked into an endless loop in which the quotient increases indefinitely.

Chapter 4

APPLICATIONS AND MATHEMATICAL MODELS

4-1. Introduction

Mathematics does not deal directly with physical objects. Mathematics does talk about ideas such as points, lines, numbers, and functions. If these two statements are both true, then how can you use mathematics to solve problems dealing with real life objects or situations? The following story, told by a mathematician, may help you see the answer to this question.

> "On a recent trip to New York, I got off at Pennsylvania Station. I walked to the taxi platform, but the train had been crowded and soon dozens of people poured out of the station for cabs. Some of them waved and yelled; some stepped in front of the rest; others, who had employed porters, seemed to be getting special treatment. I waited on the curb, convinced that if I acted in a civilized, proper manner I would attract a cab driver. However, when a driver stopped, he was snatched out from under me. I finally gave up, took the subway, denouncing the railroad, the cabbies, and people in general, and hoping that they would all get stuck for hours in the crosstown traffic.
>
> "Several weeks later, on a trip to Philadelphia, I got off at the 30th St. Station. I walked to the taxi platform. A sign told me to take a number. A man in charge called the cabs in numerical order, and I was soon on my way to the hotel. It was quick, it was pleasant, it was civilized. This was an example of a subtle, though very simple way in which mathematics can affect social affairs. The rule for loading was the order of arrival on the platform. The numbers have an order, and were used as a model of people standing in line without the inconvenience and indignity of standing in a line. The numbers were used to help turn raving madmen into polite human beings."[1]

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[1] From an address delivered by Philip J. Davis at the Annual Dinner of the Society of the Sigma Xi, Auburn University, Auburn, Alabama, May 23, 1962.

One way in which mathematics is used in the solution of real world problems is indicated in the above story. In order to solve problems about real life objects we usually create a "Mathematical Model" in which the real life objects are represented as mathematical objects. It is the purpose of this chapter to review and present some ways in which mathematical models play a part in applying mathematical ideas to the real world.

Let us begin with the uses made of the so-called <u>natural numbers</u>: 1, 2, 3, ... These numbers are also called <u>counting numbers</u>. You can easily imagine situations in which even quite primitive peoples would have a need for something like counting. It is also easy to imagine that once this dive into mathematics had been taken such things as "addition" of counting numbers would be invented to describe something about what happens when two sets of things are combined and that some early mathematical genius might discover, for example, the commutative property of addition of counting numbers. Our main point here is that what you have done for some time in applying numbers and geometry is not very different in basic spirit and method from what is done by men who write about how to use mathematical models to solve problems in business or an Einstein who seeks mathematical models for the functioning of things in the universe. (Working out mathematical models for such problems usually requires more mathematical knowledge than you have now, of course.)

With these things in mind, please consider the following exercises.

<u>Exercises [illegible]</u>

(Class Discussion)

In each of the following exercises you are given a mathematical model. Try to imagine and describe a real situation that can be represented by the given mathematical model.

<u>Example</u>: Given the set of natural numbers and the operation of addition. Describe a situation that is represented by the above model.

<u>Answer</u>: The total number of students in a mathematics classroom where there are 15 boys and 14 girls.

(The specific mathematical model of this situation would be (15 + 14).)

1. Give examples of situations in the world for which the mathematical model produced in the world of mathematics involves addition, or multiplication, the set of natural numbers, and one of the relations =, <, or >.

2. Give a situation using addition where the numbers that represent things in the real world are natural numbers but where actual counting would [illegible] possible.

3. Give an example of a situation in the world for which the mathematical model would involve fairly small natural numbers and subtraction. Now think of a different kind of situation for which the mathematical model would be exactly the same.

4. Give an example of a situation for which the mathematical model would [illegible] and division. Now try to think of a quite [illegible] which would have the same mathematical model.

5. [illegible] which would be described by natural numbers in the [illegible] counting of individual objects probably did not [illegible]

6. [illegible] where the initial description is in terms of natural [illegible] in the world of mathematics forces one [illegible]

7. [illegible] appropriate mathematical description [illegible]

8. [illegible] the mathematical model [illegible] and arithmetic.

9. [illegible] appropriate mathematical model involves [illegible]

[illegible] situations by numbers is that numbers [illegible] to move around from one place to [illegible] Numerical information can be sent over telephone [illegible] on paper, or easily "fed into" a computer. [illegible] that the same numerical expression might serve as a [illegible] different kinds of situations in the real world.

In the first part of this section you found a situation or problem which could be represented by a given mathematical model. In the following exercises, you will be given a problem situation and asked to suggest a mathematical model which could represent the situation.

Exercises 4-1b

(Class Discussion)

1. Suppose that you're standing on a spot in a room and could shoot anybody in the room with your water pistol. Are there any room shapes which would prevent you from doing this?

2. Suppose that you are the manager of a baseball team. You need a new shortstop. You can trade for Willie Much or Mickey Little, both of whom appear to be equally good glove men. In previous play Willie Much has come to bat 225 times and has 53 hits. Mickey Little has come to bat 183 times and has 43 hits. On the basis of this information which would you choose?

3. A T.V. antenna wire enters a room at one of the corners formed by two walls and the ceiling. The owner of the house wanted the wire to run down through a wall and under the floor to the opposite corner formed by two walls and the floor. What is the shortest length of wire he can use?

4. Suppose that you go to a picnic and are invited to join either of two tables. At table A there are now sitting 7 people with 5 quarts of ice cream. At table B there are sitting 10 people with 7 quarts of ice cream. At which table will your share of ice cream be greater?

The previous exercises may have given you some understanding of the reasons why people are encouraged to learn a good deal of mathematics. It turns out that mathematics is not only convenient, but very useful in dealing with real world events and problems. This is true even though mathematics itself is made up of things that are not "real" at all, at least not in the sense that molecules, cows, bacertia, rockets, bridges, etc., are "real". The discovery of a good mathematical model for a given situation is an interesting but challenging process.

4-2. Situations Leading to Geometric Models

In Chapter 1 you represented mathematical ideas, such as lines, rays, planes, and angles, with drawings that were referred to usually as geometric figures. Such figures and their characteristics serve as a rich source of mathematical models of real life situations.

In Chapter 2, you considered the problem of falling objects and Galileo's experiment. Now we want to take a look at the modeling process involved in these chapters.

We made the following assumptions when talking about Galileo's experiment:

(1) We thought of the falling objects as points. (More accurately we thought of the locations of the objects as being points.)

(2) We thought of the surface of the earth as a plane, and we thought of the paths of the objects as being parallel line segments, both perpendicular to the earth's surface.

(3) We neglected air resistance, that is, we said the objects were falling in a vacuum, (no air).

(4) We assumed that the height that an object is dropped from has no effect on how far it travels in one second.

(5) Finally we assumed that the distance traveled by a falling object is given by the formula:

$$d = 16t^2$$

where t is the time in seconds and d is the distance in feet.

This is a rather strange picture of the world! The earth is a plane with no air above it and a falling object is squeezed down into a single point.

In fact, every one of our assumptions is wrong. We know that the earth is roughly spherical in shape and that falling objects will fall toward the center of the earth and their paths will not be parallel.

Furthermore, the distance traveled by a falling body in one second is not independent of the height of the starting point. Even if we neglect the effect of air resistance, an object falling from a mile high will fall less far in a second than an object dropped near the earth's surface. The amount less would be about one part in 2000. (The change in the pull of gravity due to the distance above the earth is responsible for the difference mentioned here.)

Air resistance is certainly not always negligible. It is because of air resistance that a piece of paper falls more slowly than a penny.

All of these remarks must have weakened your confidence in our model. That was what they were suppose to do. Now we are going to rebuild your confidence.

Although the earth is a sphere, it is such a big sphere that a small portion of its surface is very nearly a plane. If two objects are dropped so that they land no more than 100 feet apart, then their paths miss being parallel by about $\frac{1}{3500}$ of one degree, which is practically negligible.

The effect of the height of the starting point only produces a difference of one part in 2000 for objects dropped from a mile high. The effect will be even more negligible if we consider only objects dropped from within a few hundred feet of the earth's surface.

The effect of air resistance is very complicated. Its effect depends on the weight, shape, and speed of the falling body. For objects that are nearly round, fairly heavy, and falling for no more than two or three seconds (so as not to build up too much speed), we can comfortably neglect air resistance.

So our model for the motion of falling bodies is not so bad after all. In fact this model is used for very accurate scientific calculations involving problems where the distance above the earth is relatively short. In such work, however, the more precise formula $d = 16.1t^2$ is used instead of $d = 16t^2$.

Exercises 4-2

(Class Discussion)

The following exercises give some situations which lead to geometric models. Try to describe these models and, if possible, indicate what assumptions about the real world you have made to get your model.

1. On a shelf in a market stand two cans of beans. The first is twice as tall as the second, but the second has a radius twice that of the first. If the second can costs twice as much as the first, which is the better buy?

2. Napoleon's forces, marching into enemy territory, came upon a river whose width they did not know. Napoleon demanded of his officers the width of the river. A young officer immediately stood erect on the bank and pulled the visor of his cap down over his eyes until his line of vision was on the edge of the opposite shore. He then turned and sighted along the shore and noted the point where his visor rested. He then paced off this distance along the shore. Why was this distance that he paced off an approximation of the width of the river?

3. In book-binding a large sheet is usually printed so that after folding and cutting, the pages appear in proper position.

   (a) Suppose that a large rectangular sheet is to be folded, left-right, and bottom-top, forming an 8-page section. Determine, before folding, the proper numbering of the pages.

   (b) How should the position of the print on each page be located so that when folded it's not upside down?

4. How many square inches of skin do you have?

4-3. How Do You Pack Your Marbles?

Recently a company marketed a salt with grains that were diamond shaped crystals instead of just ordinary cubes. The claim, which appears to be true, is that the grains, shaped like diamonds, do not "bounce" off your food like ordinary salt does. However, it turns out that it takes a larger box to pack the same weight of the diamond shaped salt compared to the space needed for the same weight of ordinary salt. This means that cooks using this salt must increase the amount used by about $\frac{1}{3}$ in order to have the same amount of salt that they would have used if they had used ordinary salt. (Have you ever tried to measure out $\frac{1}{3}$ of $\frac{1}{2}$ of a teaspoon of salt?)

The situation described above introduces a problem which has some interesting applications. In the design of insulating materials one is interested in having air space in the form of small "pockets" of air which are not large enough to permit circulation. One way to simplify such problems is to create a mathematical model. That is, to consider the packing of small spheres, like marbles, between two layers of hard material. Sometimes the surface area of the spheres must be taken into account, as well as the physical properties of the materials themselves. Similar problems occur in the design and testing of plastics.

Suppose that you have a large number of perfectly spherical marbles which you want to pack into a very large barrel. How should you pack the marbles so that you get in as many as possible?

The following exercises will develop an answer to this question and illustrate some of the important procedures involved in creating mathematical models.

Exercises 4-3

(Class Discussion)

1. Our first simplifying assumption was contained in the statement that we had a "very large barrel". So now we are really just asking, "How do you pack the marbles so that you have the greatest number marbles per cubic foot, in, for example, a box?" We still need to do more in order to bring the problem down to a level where there's some hope of solving it. We need to make some additional simplifying assumptions.

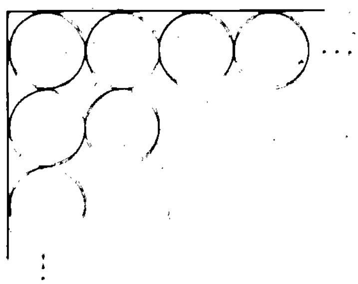
(a) Figures in three dimensional space are sometimes complicated to think about. What figure, in the plane, is related to a box?

(b) What figure, in the plane, is related to the spherical marbles?

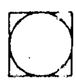
(c) State a problem similar to the "marble-barrel" problem using the figures suggested in parts (a) and (b) above.

2. Suppose we use pennies as identical circular disks and see how many we can place side by side, without overlapping in a given plane region.

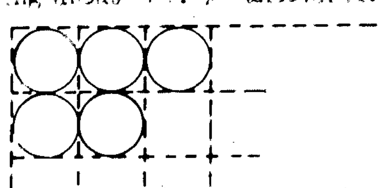
(a) If you arrange the pennies in a square in the following way, how many do you think you could pack in the square?



Assume that the diameter (the distance across) of the penny is the unit of distance, and that the square is 8 units on a side.

(b) This method really amounts to thinking of each circle as inscribed in a square one unit on a side, like this, 

and then fitting squares together so that their interiors fill all the square region. (They actually cover about 78.5% of the region, leaving about 21.5% uncovered.)

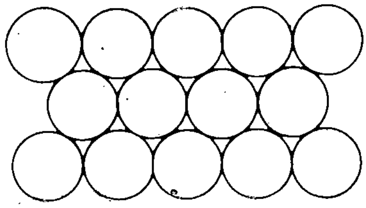


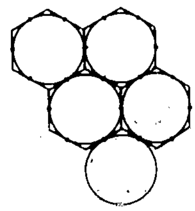
How many pennies does each penny in the middle of the arrangement touch?

(c) Can you think of another way of fitting the pennies together so that the "inner" pennies will touch more than four pennies? Draw a picture illustrating your method. How many other pennies do the "inner" pennies touch?

(d) How many pennies can you put into your 8 unit square with your new "packing" method?

Probably the following arrangement occurred to you as a good possibility.



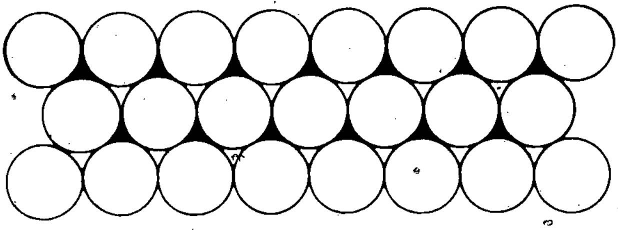


It is helpful to think of each disk as being inscribed in a regular hexagon and the hexagons fitted together as shown by the dotted lines in the figure to the right. Hexagons will fit together to cover a plane region without overlapping. In this arrangement some 90.7% of the plane region is covered by the disks, which is far better than our previous arrangement.

Can you think of other arrangements of the pennies which might be better than this? Actually the arrangement above can be shown to be the best one, though we will not try to give a proof of this fact.

Now let us return to the problem of packing the marbles in the barrel. Do you have a guess about the best way to pack them? Experiment with some marbles before you go on and see if you can make a guess as to the most efficient packing.

One possible procedure would be to start by putting a layer of marbles with their centers all in a plane parallel to the bottom--that is, a layer on the bottom of the box or barrel or whatever we are filling. From the top, they will look just like a covering of a plane region with circles, and in the light of our discussion about circles it seems that we should arrange our spheres in the same way, as shown below.



Now it seems plausible that we should try to make a second layer of marbles. Do you think it would be a good idea to place this layer of marbles such that each marble is directly above one in the first layer? No, it appears that we would get a better packing by trying to place the marbles of the second layer over the "pockets" or "holes" in the first layer. Actually there is not room to put a marble above each hole, but we can place a layer above half of the holes, say those shaded in the figure above. Then a third layer can be placed on the second, covering half of the "holes" in the second layer. This can be done in two ways, depending on which set of "holes" we choose to fill. The spheres of the third layer may be exactly above those of the first or may be above the unshaded "holes" in the first layer.

This seems to be a reasonable guess about the best possible packing. It can be shown that for this method of packing the ratio of the volume of the marbles to that of the region is $\frac{\pi}{3\sqrt{2}} \approx 0.7405$, so this packing fills about 74% of the space with spheres. No one knows whether or not this is really the best packing. The best result known so far was obtained by a British mathematician, Rankin, in 1947, who showed that there is no packing in which the spheres fill more than 82.8% of the volume of space.

4-4. Some Other Mathematical Models You Have Known

In the first three chapters you developed several very useful ideas which will help you create good mathematical models of real life situations. The obvious ones are the ideas about points, lines, planes, tables, graphs, functions, algorithms, and flow charts. These ideas, along with your background in arithmetic and measurement, make up a powerful set of tools that you can use to investigate many significant problem situations. As you proceed through your mathematics courses you will continually expand and

refine these and other important ideas. This should enable you to think about an even greater variety of practical situations.

The following exercises illustrate the use of some of the mathematical ideas you have developed, in the construction of appropriate models for some different situations.

Exercises 4-4

(Class Discussion)

1. One of the interesting problems facing those who design computers is how to design a computer that will translate a foreign language into English. Let's "tackle" a simpler problem. Suppose you are asked to find a step-by-step procedure (an algorithm) which will translate Roman numerals into ordinary Arabic numerals, (say numerals whose values are less than or equal to 1000), can you write such an algorithm? The following table shows letters used by the Romans to write their numerals.

Arabic numeral	1	5	10	50	100	500	1000
Roman numeral	I	V	X	L	C	D	M

The values of the Roman symbols are added when a symbol representing a larger quantity is placed to the left in the numeral.

$$MDCLXVI = 1000 + 500 + 100 + 50 + 10 + 5 + 1 = 1666.$$

$$DLXI = 500 + 50 + 10 + 1 = 561.$$

When a symbol representing a smaller value is written to the left of a symbol representing a larger value, the smaller value is subtracted from the larger.

$$IX = 10 - 1 = 9.$$

$$XC = 100 - 10 = 90.$$

The Romans had restrictions on subtracting.

(1) V, L, and D (symbols representing numbers that start with 5) are never subtracted.

(2) A number may be subtracted only in the following cases:

I can be subtracted from V and X only.

X can be subtracted from L and C only.

C can be subtracted from D and M only.

Addition and subtraction can both be used to write a number. First, any number whose symbol is placed to show subtraction is subtracted from the number to its right; second, the values found by subtraction are added to all other numbers in the numeral. Notice that the table defines a function f : Roman numeral $\to$ Arabic numeral.

(a) Start at the left side of the Roman numeral and by proceeding from left-to-right, write out a basic algorithm for the translation process.

(b) Construct a flow chart for your algorithm and see if it will translate MCLIV into ordinary numerals.

(c) Do you think that something like your algorithm would work for translating a foreign language into English?

(d) What are some of the simplifying assumptions you might have to make in order to create such a model of a "translation system"?

Almost everyone is aware that any kind of work (even thinking), causes fatigue. You also know that when you get tired, you can rest awhile, recover, then go on. Suppose we want to investigate the effects of fatigue and to find out how rapidly a person recovers from physical exertion. Such studies are obviously important for biologists, physical therapists, physicians, astronauts, and the like. To see how these studies might be conducted, let us perform a simple experiment and construct a mathematical model of the situation.

An Experiment: Work in teams of three, consisting of a subject, a timer and recorder, and a counter.

The only equipment you will need is a watch with a second hand, some paper, and a pencil.

The first person will be the subject.

(1) He will use one hand in the experiment, the left hand.

(2) Sitting in a comfortable position, his arm straight in front of him resting on the desk top, fingers together, palm up, he should open and close his hand as fast as he can. He should be quite sure each time that his fingers touch the desk when the hand is open, and his fingertips touch the palm when closed.

The second person will be the timer and recorder.

(1) He will watch the second hand, start the subject, and call time at the end of 90 seconds.

(2) He will also record the total number counted by the "counter" at the end of each 90-second period.

(3) The "subject" will now be allowed to rest for 30 seconds, then start the exercise for another 90-second period. As before, the count should be recorded at the end of each 90-second period.

(4) The timer should also see that the subject's fingers straighten completely, touch the table top, and then close until the tips touch the palm.

The third person is the counter.

(1) The counter will watch carefully and count the number of times the subject's fingers touch the table top.

(2) It is important that he count quickly, but aloud, so that the timer-recorder can hear and record the count at the end of each 90-second period.

(3) To get the number of times the fingers were opened for each 90-second time period after the first period the recorder should subtract the first total count from the second total count, the second total count from the third, etc.

The following is a table of sample data. Make a table like this for each member of your team and rotate jobs until everyone has had a chance to be a subject.

Muscle Fatigue Name of Subject				
Time Period	Time in Seconds	Total Count for Left Hand	Count per Time Period for Left Hand	Difference between the count in one period and the count in the next time period
[illegible]	[illegible]	38	38	--
2nd	[illegible]	72	72 - 38 = 34	38 - 34 = 4
3rd	[illegible]	104	104 - 72 = 32	34 - 32 = 2
[illegible]	[illegible]	132	132 - 104 = 28	32 - 28 = 4
[illegible]	[illegible]	157	157 - 132 = 25	28 - 25 = 3
[illegible]	[illegible]	179	179 - 157 = 22	25 - 22 = 3
REST	30	REST	REST	
[illegible]	[illegible]	26	26	--
[illegible]	[illegible]	50	50 - 26 = 24	26 - 24 = 2
[illegible]	[illegible]	72	72 - 50 = 22	24 - 22 = 2
[illegible]	[illegible]	92	92 - 72 = 20	22 - 20 = 2
[illegible]	[illegible]	111	111 - 92 = 19	20 - 19 = 1
[illegible]	[illegible]	129	129 - 111 = 18	19 - 18 = 1

(a) Draw a graph of your data where the input is the number of the time period and the output is the "Count per Time Period".

(b) Does your graph indicate that you got tired? Did you "recover" fully in the 30-second rest period?

(c) Draw a graph of your data where the input is the number of the time period and the output is the difference between the counts in two successive periods.

(d) Can you use your mathematical model to <u>predict</u> what the "count" would have been in the 7th time period if you had <u>not</u> rested? If so, explain how you could do this.

(e) Can you use your mathematical model to <u>predict</u> what the count would have been in the 13th period if you had continued the experiment? If so, explain how you could do this.

4-4. Summary

A mathematical model tries to duplicate some of the actual characteristics of a real life situation. If these characteristics are properly represented in the model, then we can use the model to predict what might happen in different situations. To be successful a model should:

(1) contain as many of the main characteristics of the real life situation as possible;

(2) the characteristics of the real life situation that are included in the model should behave in the model like they do in the real situation; and

(3) the model should be simple enough so that the mathematical problems that are suggested by the model can be solved.

It should be clear that a mathematical model is never a perfect representation of a real life situation. Usually many "simplifying assumptions" have been made before the mathematical model is finally constructed. The answers found by solving the mathematical problems are <u>not</u> the answers to the real problem situation, but just predictions of what will be seen when the real situation is observed.